3.2 Arithmetic Sequences and Series

An arithmetic sequence is a sequence in which the next term is the previous term plus a constant.

The constant is called the common difference, written d.

Ex: Find d in the following arithmetic sequences.

a) $0.4, 0.5, 0.6, \ldots$

$d = 0.1$

b) $5, -5, -15, \ldots, -205$

$d = -10$

c) $-1, -\frac{3}{2}, -2, \ldots$

$d = -\frac{1}{2}$

Recursive Formula for Infinite Arithmetic Sequences

$$\begin{cases} a_m = \text{<insert first term here>} & \\ a_n = a_{n-1} + d & \text{for } n \geq m+1 \end{cases}$$

Ex: Write the recursive formula for
$0.4, 0.5, 0.6, \ldots$

$$\begin{cases} a_1 = 0.4 \\ a_n = a_{n-1} + 0.1 & \text{for } n \geq 2 \end{cases}$$

General Formula for Infinite Arithmetic Sequences

$$a_n = a_m + (n-m)d \quad \text{for } n \geq m$$

Ex: Find the simplified general formula
for $5, -5, -15, \ldots$

$$a_n = a_m + (n-m)d \quad \text{for } n \geq m$$

Sub $m=1$: $a_n = a_1 + (n-1)d$ for $n \geq 1$

$$a_n = 5 + (n-1)(-10) \quad \text{"}$$

$$a_n = 5 - 10n + 10 \quad \text{"}$$

$$\boxed{a_n = 15 - 10n \quad \text{for } n \geq 1}$$

Ex: An arithmetic sequence has $a_1 = 2$ and $a_{50} = 394$. Find d.

$$a_n = a_m + (n-m)d$$

$\boxed{\begin{array}{l} m = \text{first index} \\ n = \text{last index} \end{array}}$

Sub $m=1$, $n=50$:

$$a_{50} = a_1 + 49d$$
$$394 = 2 + 49d$$
$$392 = 49d$$
$$d = 8$$

Ex: An arithmetic sequence has $a_1 = 3$ and $d = 7$.

a) Find a_{21}

$$a_n = a_m + (n-m)d$$

Sub $n=21$, $m=1$:

$$a_{21} = a_1 + 20d$$
$$= 3 + 20(7)$$
$$= 143$$

b) Which term equals 717 ?

$$a_n = a_m + (n-m)d$$

Sub $m=1$, $a_n = 717$:

$$717 = a_1 + (n-1)d$$
$$717 = 3 + (n-1)(7)$$
$$714 = (n-1)(7)$$
$$102 = n-1$$
$$n = 103$$

a_{103} equals 717.

Ex: An arithmetic sequence has $a_6 = 13$ and $d = -4$. Find a_1.

$$a_n = a_m + (n-m)d$$

Sub $n=6$, $m=1$:

$$a_6 = a_1 + 5d$$
$$13 = a_1 - 20$$
$$33 = a_1$$

Ex: An arithmetic sequence has $a_6 = 21$ and $a_{18} = 45$. Find a_1 and d.

Find d:

Sub $n=18, m=6$:

$$a_n = a_m + (n-m)d$$
$$a_{18} = a_6 + 12d$$
$$45 = 21 + 12d$$
$$24 = 12d$$
$$d = 2$$

Find a_1

$$a_n = a_m + (n-m)d$$

$n=18$, $m=1$:

$$a_{18} = a_1 + 17d$$
$$45 = a_1 + 17(2)$$
$$11 = a_1$$

Arithmetic series: A sum in which the next term is the previous term plus a constant d.

An arithmetic series: $2+5+8+\dots$

Recall S_k is the sum of the first k terms.

Fact

$$\boxed{\begin{aligned} S_k &= \frac{k}{2}(a_m + a_n) \\ S_k &= \frac{k}{2}\left[2a_m + (n-m)d\right] \\ &\text{where } m = \text{~~first~~ index} \\ &\qquad n = \text{last index} \\ &\qquad k = \#\text{ terms in partial sum} \\ &\qquad k = n-m+1 \end{aligned}}$$

Ex: Consider $2+5+8+\dots$

a) Calculate S_8 directly

$S_8 = 2+5+8+11+14+17+20+23$

$= 100$

b) Calculate S_8 using $S_k = \frac{k}{2}(a_m + a_n)$

$k=8$
$m=1$;
$n=8$

$$S_8 = \frac{8}{2}(a_1 + a_8)$$
$$= 4(2+23)$$
$$= 100$$

c) Calculate S_8 using $S_k = \frac{k}{2}[2a_m + (n-m)d]$

$k=8$
$m=1$:
$d=3$

$$S_8 = \frac{8}{2}[2a_1 + 7d]$$
$$= 4[2(2) + 7(3)]$$
$$= 100$$

<u>Ex</u>: Find the sum of the first 50 terms of
$2+5+8+\dots$

Don't know a_{50}.
Use $S_k = \frac{k}{2}[2a_m + (n-m)d]$

$k=50$:
$m=1$
$n=50$

$$S_{50} = 25[2a_1 + 49d]$$
$$= 25[4 + 49(3)]$$
$$= 3775$$

Ex: Evaluate $\sum_{j=4}^{50} (6j-3)$

$$= \underset{a_m}{21} + 27 + 33 + \dots + \underset{a_n}{297}$$

$k = \#$ terms

$= n - m + 1$

$= 50 - 4 + 1$

$= 47$

$S_k = \frac{k}{2}(a_m + a_n)$

$S_{47} = \frac{47}{2}(21 + 297)$

$= 7473$