6.2 General Systems of Linear Equations

Ex: Solve

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \quad \# \\ \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 1 & 3 & 3 & 12 \\ 1 & 4 & 5 & 1 \end{array}\right] \end{array}$$

$$\begin{array}{r} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & -8 \end{array}\right]$$

$$\begin{array}{r} R_1 - 2R_2 \\ \\ R_3 - 2R_2 \end{array} \left[\begin{array}{ccc|c} & & & \\ & & & \\ 0 & 0 & 0 & -14 \end{array}\right]$$

$0x + 0y + 0z = -14$
impossible

The system of equations has no solution.

Ex: If you see a row like this, then the system has no solution

[0 0 | nonzero]

[0 0 0 | nonzero]

[all zero | nonzero]

Ex: Solve

$$\begin{array}{c} \begin{array}{ccc|c} x & y & z & \# \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 1 & 3 & 3 & 12 \\ 1 & 4 & 5 & 15 \end{array}\right] \end{array}$$

$$\begin{array}{r} \\ R_2-R_1 \\ R_3-R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \end{array}\right]$$

$$\begin{array}{r} R_1-2R_2 \\ \\ R_3-2R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Cannot make this 1
Elimination is done

Circle the leading 1 in each row

$$\begin{array}{c} \begin{array}{ccc|c} x & y & z & \# \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Can solve for x and y

z is a "free variable"

$x - 3z = 3$

$y + 2z = 3$

$$\boxed{\begin{array}{l} z = \text{any value} \\ x = 3 + 3z \\ y = 3 - 2z \end{array}}$$

Infinitely-many solutions

Ex: Find 3 solutions to the system above

$(z=0)$ $\quad x=3,\ y=3,\ z=0$

$(x,y,z)=(3,3,0)$

$(z=1)$ $\quad (x,y,z)=(6,1,1)$

$(z=-1)$ $\quad (x,y,z)=(0,5,-1)$

Ex: Solve:

a)

$$\begin{array}{c} \begin{array}{cc} x & y \end{array} \quad \# \\ \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right] \end{array} \leftarrow \text{no info}$$

$x = 4$

$y = 5$

$(x,y) = (4,5)$

b)

$$\begin{array}{c} \begin{array}{cc} x & y \end{array} \quad \# \\ \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{array}\right] \end{array}$$

no solution

Ex: Solve

$$\begin{array}{c} \begin{array}{cccc|c} w & x & y & z & \# \end{array} \\ \left[\begin{array}{cccc|c} 2 & 2 & 6 & 14 & 12 \\ 3 & 4 & 15 & 26 & 17 \end{array}\right] \end{array}$$

$R_1/2$ $$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 7 & 6 \\ 3 & 4 & 15 & 26 & 17 \end{array}\right]$$

$R_2 - 3R_1$ $$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 7 & 6 \\ 0 & 1 & 6 & 5 & -1 \end{array}\right]$$

$R_1 - R_2$ $$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 7 \\ 0 & 1 & 6 & 5 & -1 \end{array}\right]$$

Circle leading 1's in each row

Can solve for w and x

y and z are free variables

$$\boxed{\begin{array}{l} y = \text{any value} \\ z = \text{any value} \\ w = 7 + 3y - 2z \\ x = -1 - 6y - 5z \end{array}}$$

$w - 3y + 2z = 7$

$x + 6y + 5z = -1$

Ex: Solve

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \quad \# \\ \left[\begin{array}{ccc|c} 3 & 6 & 6 & 12 \\ 3 & 6 & 8 & 15 \\ 2 & 4 & 4 & 8 \end{array}\right] \end{array}$$

$$R_1/3 \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 3 & 6 & 8 & 15 \\ 2 & 4 & 4 & 8 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Can't make this 1
More further right

$$R_2/2 \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 - 2R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Circle leading 1's in each row
Can solve for x and z
y is a free variable

$x + 2y = 1$

$$\boxed{\begin{array}{l} y = \text{any value} \\ x = 1 - 2y \\ z = 3/2 \end{array}}$$

Ex: Labour Hours

	Ottoman	Sofa	Chair
Carpentry	1	3	6
Upholstery	3	6	3

380 labour hours available for Carpentry
450 " Upholstery

How many of each item to produce?
Give 3 options.

Let x = # ottomans
y = # sofas
z = # chairs

Carpentry:
Upholstery:
$$\begin{cases} x + 3y + 6z = 380 \\ 3x + 6y + 3z = 450 \end{cases}$$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \# \end{array} \\ \left[\begin{array}{ccc|c} 1 & 3 & 6 & 380 \\ 3 & 6 & 3 & 450 \end{array}\right] \end{array}$$

$$R_2 - 3R_1 \quad \left[\begin{array}{ccc|c} 1 & 3 & 6 & 380 \\ 0 & -3 & -15 & -690 \end{array}\right]$$

$$R_2/(-3) \quad \left[\begin{array}{ccc|c} 1 & 3 & 6 & 380 \\ 0 & 1 & 5 & 230 \end{array}\right]$$

$$R_1 - 3R_2 \quad \begin{array}{c} \begin{array}{ccc|c} x & y & z & \# \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -9 & -310 \\ 0 & 1 & 5 & 230 \end{array}\right] \end{array}$$

$x - 9z = -310$

$y + 5z = 230$

$$\boxed{\begin{array}{l} z = \text{any value} \\ x = -310 + 9z \\ y = 230 - 5z \end{array}}$$

Need all values ≥ 0
because these are physical objects

$$\boxed{\begin{array}{r} x \geq 0 \\ -310 + 9z \geq 0 \\ 9z \geq 310 \\ z \geq \frac{310}{9} \\ z \geq 35 \end{array}} \qquad \boxed{\begin{array}{r} y \geq 0 \\ 230 - 5z \geq 0 \\ 230 \geq 5z \\ 46 \geq z \end{array}}$$

Conclude: $35 \leq z \leq 46$

Give 3 options:

1) $z = 35$ $\quad x = 5$ $\quad y = 55$
2) $z = 40$ $\quad x = 50$ $\quad y = 30$
3) $z = 45$ $\quad x = 95$ $\quad y = 5$

etc.

Ex: A store has 3 gift items priced at \$7, \$10 and \$13. Amanda wants to buy 15 gifts and spend \$150. Give 3 options.

Let x = # \$7 items
y = \$10
z = \$13

$$\begin{cases} \# : & x+y+z=15 \\ \$: & 7x+10y+13z=150 \end{cases}$$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \quad \# \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 15 \\ 7 & 10 & 13 & 150 \end{array}\right] \end{array}$$

$$R_2 - 7R_1 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 15 \\ 0 & 3 & 6 & 45 \end{array}\right]$$

$$R_2/3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 15 \\ 0 & 1 & 2 & 15 \end{array}\right]$$

$$R_1 - R_2 \quad \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \quad \# \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 15 \end{array}\right] \end{array}$$

$x - z = 0$
$y + 2z = 15$

$$\boxed{\begin{array}{l} z = \text{any value} \\ x = z \\ y = 15 - 2z \end{array}}$$

Give 3 options (with all variables ≥ 0)

1) $z = 0$ $x = 0$ $y = 15$

2) $z = 3$ $x = 3$ $y = 9$

3) $z = 6$ $x = 6$ $y = 3$

etc.