7.2 Regular Markov Chains

A transition matrix P is *regular* if some power of P has only positive entries.
A Markov Chain is a *regular Markov Chain* if its transition matrix is regular.

Ex: Is the Markov Chain regular?

a) $P = \begin{bmatrix} 0.3 & 0.7 \\ 0.9 & 0.1 \end{bmatrix}$ all positive ✓

Yes

b) $P = \begin{bmatrix} 0 & 1 \\ 0.3 & 0.7 \end{bmatrix}$

$$P^2 = PP = \begin{bmatrix} 0 & 1 \\ 0.3 & 0.7 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.7 \\ 0.21 & 0.79 \end{bmatrix}$$

Yes

c) $P = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}$

$$P^2 = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.75 & 0.25 \end{bmatrix}$$

$$P^3 = P^2 P = \begin{bmatrix} 1 & 0 \\ \# & \# \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \# & \# \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \# & \# \end{bmatrix}$$

Frist row of P^k will always have a 0

No

Ex: CleanHair Shampoo Customers

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

2 observations:

(1) $$\begin{bmatrix} 0.667 & 0.333 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.667 & 0.333 \end{bmatrix}$$

Next state equals the current state

(2) Say $S_0 = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$

$$S_1 = S_0 P = \begin{bmatrix} 0.48 & 0.52 \end{bmatrix}$$

$$\vdots$$

$$S_8 = \begin{bmatrix} 0.651 & 0.349 \end{bmatrix}$$

$$\vdots$$

As $k \to \infty$, $S_k \approx \begin{bmatrix} 0.667 & 0.333 \end{bmatrix}$

The state matrix S is called a <u>stationary matrix</u> if $SP = S$

FACT

Let P be the transition matrix for a <u>regular</u> Markov chain. Then:

1) There is a stationary matrix S
2) As $k \to \infty$, $S_k \approx S$

<u>EX</u>: $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$

Find the stationary matrix

Let $S = [s_1 \quad s_2]$

$$SP = S$$

$$[s_1 \quad s_2] \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = [s_1 \quad s_2]$$

$$[0.9s_1 + 0.2s_2 \quad 0.1s_1 + 0.8s_2] = [s_1 \quad s_2]$$

$0.9s_1 + 0.2s_2 = s_1 \rightarrow -0.1s_1 + 0.2s_2 = 0$ ①

$0.1s_1 + 0.8s_2 = s_2 \rightarrow 0.1s_1 - 0.2s_2 = 0$ ②

Not enough info

$s_1 + s_2 = 1$ ③

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} s_1 & s_2 & \# \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -0.1 & 0.2 & 0 \\ 0.1 & -0.2 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ R_2+0.1R_1 \\ R_3-0.1R_1 \end{array}\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0.3 & 0.1 \\ 0 & -0.3 & -0.1 \end{array}\right]$$

$$\begin{array}{c} \\ R_2/0.3 \\ \\ \end{array}\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1/3 \\ 0 & -0.3 & -0.1 \end{array}\right] \leftarrow \frac{0.1}{0.3} = \frac{1}{3}$$

$$\begin{array}{c|cc|c} & s_1 & s_2 & \# \\ R_1-R_2 & & & \\ & & & \\ R_3+0.3R_2 & & & \end{array}\left[\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{array}\right]$$

$s_1 = 2/3$

$s_2 = 1/3$

$$S = \left[\frac{2}{3} \quad \frac{1}{3}\right]$$

Check: $SP = S$ ✓

Ex: $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.9 & 0.1 & 0 \end{bmatrix}$

a) Find the stationary matrix

$\boxed{\text{Solve } SP = S \text{ together with } s_1 + s_2 + s_3 = 1}$

$SP = S$

$$[s_1\ s_2\ s_3] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.9 & 0.1 & 0 \end{bmatrix} = [s_1\ s_2\ s_3]$$

$$[0.9s_3 \quad s_1 + 0.1s_3 \quad s_2] = [s_1 \quad s_2 \quad s_3]$$

$0.9s_3 = s_1 \rightarrow$
$s_1 + 0.1s_3 = s_2 \rightarrow$
$s_2 = s_3 \rightarrow$

$$\begin{cases} -s_1 + 0.9s_3 = 0 & (1) \\ s_1 - s_2 + 0.1s_3 = 0 & (2) \\ s_2 - s_3 = 0 & (3) \end{cases}$$

Not enough info

$s_1 + s_2 + s_3 = 1$ (4)

$$\begin{array}{cccc} s_1 & s_2 & s_3 & \# \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -1 & 0 & 0.9 & 0 \\ 1 & -1 & 0.1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1.9 & 1 \\ 0 & -2 & -0.9 & -1 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$
\begin{array}{l}
R_1 - R_2 \\
\\
R_3 + 2R_2 \\
R_4 - R_2
\end{array}
\left[\begin{array}{ccc|c}
1 & 0 & -0.9 & 0 \\
0 & 1 & 1.9 & 1 \\
0 & 0 & 2.9 & 1 \\
0 & 0 & -2.9 & -1
\end{array}\right]
$$

$$
\begin{array}{l}
\\
\\
R_3/2.9 \\
\\
\end{array}
\left[\begin{array}{ccc|c}
1 & 0 & -0.9 & 0 \\
0 & 1 & 1.9 & 1 \\
0 & 0 & 1 & \frac{10}{29} \\
0 & 0 & -2.9 & -1
\end{array}\right] \leftarrow \frac{1}{2.9} = \frac{10}{29}
$$

$$
\begin{array}{l}
\\
R_1 + 0.9R_3 \\
R_2 - 1.9R_3 \\
\\
R_4 + 2.9R_3
\end{array}
\begin{array}{c}
\begin{array}{cccc} s_1 & s_2 & s_3 & \# \end{array} \\
\left[\begin{array}{ccc|c}
1 & 0 & 0 & \frac{9}{29} \\
0 & 1 & 0 & 10/29 \\
0 & 0 & 1 & 10/29 \\
0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{l}
\\
\leftarrow 0.9\left(\frac{10}{29}\right) = \frac{9}{29} \\
\leftarrow 1 - 1.9\left(\frac{10}{29}\right) = \frac{29}{29} - \frac{19}{29} \\
\\
\\
\end{array}
$$

$$s_1 = \frac{9}{29}$$

$$s_2 = \frac{10}{29}$$

$$s_3 = \frac{10}{29}$$

$$S = \left[\frac{9}{29} \quad \frac{10}{29} \quad \frac{10}{29}\right]$$

b) As $k \to \infty$ what does S_k look like?

As $k \to \infty$ $\quad S_k \to S$

~~c) Find the limiting matrix $\bar{P}$~~

Ignore this part

$$\cancel{\bar{P} = \begin{bmatrix} S \\ S \\ S \end{bmatrix} = \begin{bmatrix} \frac{9}{29} & \frac{10}{29} & \frac{10}{29} \\ \frac{9}{29} & \frac{10}{29} & \frac{10}{29} \\ \frac{9}{29} & \frac{10}{29} & \frac{10}{29} \end{bmatrix}}$$