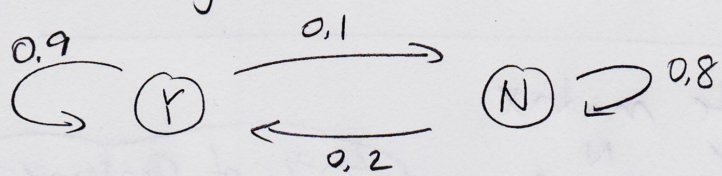
# 7.1 Properties of Markov Chains

Ex: CleanHair Shampoo Company

Y = Consumer uses CleanHair
N = " another brand

Transition diagram:




90% of CleanHair customers will buy it again
10% " won't

20% of other brand customers will buy CleanHair next time
80% " won't

Transition matrix:

$$P = \begin{array}{c c} & \begin{array}{cc} Y & N \end{array} \leftarrow \text{next state} \\ \begin{array}{c} Y \\ N \end{array} & \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$

current state

Def

$\boxed{\text{Markov Chain: Sequence of trials where the probability of moving between states is given by } P.}$

Properties of a Transition Matrix:

1) It's a square matrix
2) All entries must be $\geq 0$
3) Each row sums to 1

Note: In a transition diagram, sum of arrows leaving each state is 1.

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Initial-state matrix

$$S_0 = \begin{array}{cc} Y & N \\ \left[\, 0.4 \right. & \left. 0.6 \,\right] \end{array}$$

40% of consumers use CleanHair now

First-state matrix

$$S_1 = S_0 P$$

$$= [0.4 \ \ 0.6] \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{array}{cc} Y & N \\ \left[\, 0.48 \right. & \left. 0.52 \,\right] \end{array}$$

percentages for next purchase

Second-state matrix

$$S_2 = S_1 P$$

$$= [0.48 \ \ 0.52] \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{array}{cc} Y & N \\ \left[\, 0.536 \right. & \left. 0.464 \,\right] \end{array}$$

percentages for 2 purchases from now

$$\boxed{\text{In general: } k^{th}\text{-state matrix} \quad S_k = S_{k-1} P}$$

$$\boxed{\begin{array}{l}\text{FACT} \\ \text{In any state matrix, the entries sum to 1.}\end{array}}$$

Notice $S_1 = S_0 P$

$$\begin{aligned} S_2 &= S_1 P \\ &= (S_0 P) P \\ &= S_0 P^2 \end{aligned}$$

$$S_3 = S_0 P^3$$

$$\boxed{\text{FACT} \qquad S_k = S_0 P^k}$$

Ex: Given $P^2 = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{cc} A & B \\ \hline \end{array}$ $\begin{array}{c|cc} & A & B \\ \hline A & 1 & 0 \\ B & 0.75 & 0.25 \end{array}$ and $P^6 = \begin{array}{c|cc} & A & B \\ \hline A & 1 & 0 \\ B & 0.9844 & 0.0156 \end{array}$

What is the probability of going from B to A in 2 trials? 6 trials?

In 2 trials: use $P^2$

$$P^2 = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} & \\ * & \end{bmatrix} \end{array}$$

Probability = 0.75

In 6 trials: use $P^6$

$$P^6 = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} & \\ * & \end{bmatrix} \end{array}$$

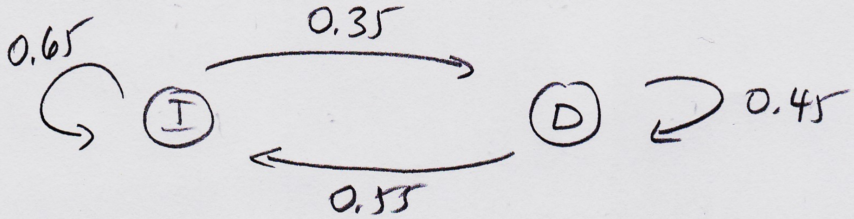
Probability = 0.9844

Ex: Stock prices

If price increases one day, the probability that it increases the next day is 0.65

" decreases " decreases " 0.45

a) Draw a transition diagram




b) Find the transition matrix

$$P = \begin{array}{c c} & \begin{array}{cc} I & D \end{array} \\ \begin{array}{c} I \\ D \end{array} & \begin{bmatrix} 0.65 & 0.35 \\ 0.55 & 0.45 \end{bmatrix} \end{array}$$ ← next day

↑
current day

c) There is an 80% probability that stock XYZ's price will decrease today.
Pr( it increases 2 days from now)?

$$S_0 = \begin{array}{cc} I & D \\ [\,0.2 & 0.8\,] \end{array}$$

$\boxed{\text{Want } S_2 = S_0 P^2}$

$$P^2 = PP = \begin{bmatrix} 0.65 & 0.35 \\ 0.55 & 0.45 \end{bmatrix} \begin{bmatrix} 0.65 & 0.35 \\ 0.55 & 0.45 \end{bmatrix} = \begin{bmatrix} 0.615 & 0.385 \\ 0.605 & 0.395 \end{bmatrix}$$

$$S_2 = S_0 P^2 = \begin{bmatrix} 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.615 & 0.385 \\ 0.605 & 0.395 \end{bmatrix}$$

$$\begin{array}{cc} I & D \end{array}$$

$$= \begin{bmatrix} 0.607 & 0.393 \end{bmatrix}$$

$$\boxed{60.7\%}$$