6.5 Gauss-Jordan Method for Finding A^{-1}

$[A \mid I] \rightsquigarrow [I \mid A^{-1}]$ using 3 operations:

1) Swap 2 rows
2) Multiply/divide a row by a nonzero #
3) current row ± # (pivot row)

Notes: 1) Works for 2x2, 3x3 etc.
2) If we see a row of 0's then A^{-1} does not exist

Ex: Find A^{-1} for $A = \begin{bmatrix} 2 & 10 & 2 \\ 0 & 4 & 1 \\ 2 & 14 & 2 \end{bmatrix}$

$[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 2 & 10 & 2 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 \\ 2 & 14 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_1/2 \quad \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1/2 & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 \\ 2 & 14 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 - 2R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1/2 & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 \\ 0 & 4 & 0 & -1 & 0 & 1 \end{array}\right]$$

$R_2/4$ $\left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & 1/4 & 0 & 1/4 & 0 \\ 0 & 4 & 0 & -1 & 0 & 1 \end{array}\right]$

$R_1 - 5R_2$, $R_3 - 4R_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -1/4 & 1/2 & -5/4 & 0 \\ 0 & 1 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right]$

$R_3/(-1)$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -1/4 & 1/2 & -5/4 & 0 \\ 0 & 1 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$

$R_1 + \frac{1}{4}R_3$, $R_2 - \frac{1}{4}R_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1 & -1/4 \\ 0 & 1 & 0 & -1/4 & 0 & 1/4 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$

$1/2 + \frac{1}{4} = 3/4$

$= A^{-1}$

$[I \mid A^{-1}]$

Ex: Find A^{-1} for $A = \begin{bmatrix} 1 & 1 & 6 \\ 0 & 2 & 3 \\ 2 & 2 & 12 \end{bmatrix}$

$[A \mid I]$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 6 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 12 & 0 & 0 & 1 \end{array}\right]$

$R_3 - 2R_1$ $\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & -2 & 0 & 1 \end{array}\right]$

Row of 0's
A^{-1} does not exist

Ex: Find A^{-1} and use it to solve

$$\begin{cases} 5x + 6y + 7z = 56 \\ x + y + z = 9 \\ x + z = 6 \end{cases}$$

$$A = \begin{bmatrix} 5 & 6 & 7 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 5 & 6 & 7 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_3$ $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 5 & 6 & 7 & 1 & 0 & 0 \end{array}\right]$$

$R_2 - R_1$, $R_3 - 5R_1$ $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 6 & 2 & 1 & 0 & -5 \end{array}\right]$$

$R_3 - 6R_2$ $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 & -6 & 1 \end{array}\right]$$

$R_3/2$ $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1/2 & -3 & \frac{1}{2} \end{array}\right]$$

$R_1 - R_3$ $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 3 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1/2 & -3 & \frac{1}{2} \end{array}\right] = A^{-1}$$

$[I \mid A^{-1}]$

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/2 & 3 & 1/2 \\ 0 & 1 & -1 \\ 1/2 & -3 & 1/2 \end{bmatrix} \begin{bmatrix} 56 \\ 9 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$