6.4 The Inverse of a Matrix

Recall $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ etc.

DEF

For a square matrix A (2×2, 3×3 etc.)
A^{-1} has the property that:

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

A^{-1} is called the inverse of A

Note: Some square matrices don't have an inverse.

Ex: $A = \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix}$

Check that $\begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix} = A^{-1}$

→ Check $AA^{-1} = I$ or $A^{-1}A = I$
Only need to check one.

$$AA^{-1} = \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix}\begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I \quad \checkmark$$

FACT

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the <u>determinant of A</u> is $D = ad - bc$

$$A^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

<u>Notes</u>: 1) If $D = 0$ then A^{-1} does not exist

2) Formula only works for 2x2 matrices

<u>Ex</u>: Find A^{-1} for $A = \begin{bmatrix} -5 & -2 \\ 2 & 1 \end{bmatrix}$

$D = -5(1) - (-2)(2) = -1$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 5 \end{bmatrix}$$

Check: $AA^{-1} = \begin{bmatrix} -5 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ —

Recall $\begin{cases} x+2y=7 \\ 3x+4y=15 \end{cases}$

Matrix equation: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \end{bmatrix}$

Symbolically: $AX=B$

Want to solve $AX=B$:

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

FACT

A system of equations $AX=B$ has solution $X=A^{-1}B$ (if A^{-1} exists)

Ex: Solve by finding A^{-1}

$$\begin{cases} 7x-3y=25 \\ 2x+y=9 \end{cases}$$

$$A = \begin{bmatrix} 7 & -3 \\ 2 & 1 \end{bmatrix}$$

$$D = 7(1)-(-3)(2) = 13$$

$$A^{-1} = \frac{1}{13}\begin{bmatrix} 1 & 3 \\ -2 & 7 \end{bmatrix}$$

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 1 & 3 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 25 \\ 9 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} 52 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$(x, y) = (4, 1)$