6.1 Systems of Linear Equations with Unique Solution

System of equation:

$$\begin{cases} 3x + 6y = 30 \\ 2x + 8y = 32 \end{cases}$$

In matrix form:

$$\begin{matrix} x & y & \# \end{matrix}$$
$$\left[\begin{array}{cc|c} 3 & 6 & 30 \\ 2 & 8 & 32 \end{array}\right]$$

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3 Elementary Row Operation:

1) Swap 2 rows
2) Multiply/divide a row by a nonzero #
3) Current row - # (pivot row)

Aim to get $\left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \end{array}\right]$ or $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array}\right]$

"diagonal form"

Ex: Solve $\begin{cases} 3x + 6y = 30 \\ 2x + 8y = 32 \end{cases}$

$$\left[\begin{array}{cc|c} x & y & \# \\ 3 & 6 & 30 \\ 2 & 8 & 32 \end{array}\right]$$

Get a 1 "the pivot"

$R_1/3$ $\left[\begin{array}{cc|c} \textcircled{1} & 2 & 10 \\ 2 & 8 & 32 \end{array}\right]$

Get 0's in rest of column

$\left[\begin{array}{cc|c} 1 & 2 & 10 \\ 0 & 4 & 12 \end{array}\right]$

$R_2 - 2R_1$

row ± # (pivot row)

Get a 1

$R_2/4$ $\left[\begin{array}{cc|c} 1 & 2 & 10 \\ 0 & \textcircled{1} & 3 \end{array}\right]$

Get 0's in rest of column

$R_1 - 2R_2$ $\left[\begin{array}{cc|c} x & y & \# \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{array}\right]$

$x = 4$

$y = 3$

$(x, y) = (4, 3)$ ✓

"Gauss-Jordan Elimination"

Ex: Solve $\begin{cases} -x+y = -1 \\ x+z = 4 \\ 6x-3y+2z = 10 \end{cases}$

$$\left[\begin{array}{ccc|c} x & y & z & \# \\ -1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 4 \\ 6 & -3 & 2 & 10 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} \textcircled{1} & 0 & 1 & 4 \\ -1 & 1 & 0 & -1 \\ 6 & -3 & 2 & 10 \end{array}\right]$$

R_2+R_1, R_3-6R_1

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & \textcircled{1} & 1 & 3 \\ 0 & -3 & -4 & -14 \end{array}\right]$$

R_3+3R_2

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -5 \end{array}\right]$$

$R_3/(-1)$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & \textcircled{1} & 5 \end{array}\right]$$

R_1-R_3, R_2-R_3

$$\left[\begin{array}{ccc|c} x & y & z & \# \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$(x, y, z) = (-1, -2, 5)$

Ex: Solve $\begin{cases} x - 3y + 4z = 1 \\ 4x - 10y + 10z = 4 \\ -3x + 9y - 5z = -6 \end{cases}$

$$\begin{array}{c} \\ R_2 - 4R_1 \\ R_3 + 3R_1 \end{array} \left[\begin{array}{ccc|c} x & y & z & \# \\ \hline \textcircled{1} & -3 & 4 & 1 \\ 4 & -10 & 10 & 4 \\ -3 & 9 & -5 & -6 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -3 & 4 & 1 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 7 & -3 \end{array}\right]$$

$$R_2/2 \quad \left[\begin{array}{ccc|c} 1 & -3 & 4 & 1 \\ 0 & \textcircled{1} & -3 & 0 \\ 0 & 0 & 7 & -3 \end{array}\right]$$

$$R_1 + 3R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 7 & -3 \end{array}\right]$$

$$R_3/7 \quad \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & \textcircled{1} & -\frac{3}{7} \end{array}\right]$$

$R_1 + 5R_3$
$R_2 + 3R_3$

$$\begin{array}{ccc|c} x & y & z & \# \\ \hline 1 & 0 & 0 & -\frac{8}{7} \\ 0 & 1 & 0 & -\frac{9}{7} \\ 0 & 0 & 1 & -\frac{3}{7} \end{array}$$

$1 + 5\left(-\frac{3}{7}\right) = \frac{7}{7} - \frac{15}{7}$

$0 + 3\left(-\frac{3}{7}\right)$

$$(x, y, z) = \left(-\frac{8}{7}, -\frac{9}{7}, -\frac{3}{7}\right)$$

Ex: Each hat takes 3 hours and \$2 to produce
Each coat 6 hours \$8
Your company has 30 hours and \$32 available each day.
How many hats and coats should you produce?

Let x = # hats
y = # coats

hours: $3x + 6y = 30$
(hats, coats)

\$: $2x + 8y = 32$

$$\begin{cases} 3x + 6y = 30 \\ 2x + 8y = 32 \end{cases}$$

$(x, y) = (4, 3)$ from Example 1