Camosun College
Dept. of Mathematics & Statistics
Math 251 Skeleton Notes

The section numbering, section titles and notation follow the textbook *Linear Algebra: A Modern Introduction, Third Edition* by David Poole. All examples were created by Leah Howard. These lecture notes conform to the Fair Dealing Policy Guidelines in the Copyright Act.

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Course Overview

Matrix Algebra is also known as “Linear Algebra” or “Algebra and Geometry.”

A geometry problem could involve visualizing lines and planes in 3D space.

An algebra problem could involve calculating distances and angles, especially in higher dimensions.

Many problems in Matrix Algebra involve the interplay of geometry and algebra.

Why do we need higher dimensions? Tracking an object’s spatial location and temperature is a 4D problem.

Chapter 1: Vectors

1.1 The Geometry and Algebra of Vectors

Definition: A **vector** is a line segment with direction. Used for velocity, forces etc.

Example: Given $O = (0,0), A = (4,2)$ and $B = (5,5)$. Draw the vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$. Then write them in component notation.

Example: Given $C = (-1, -3)$ and $D = (2, -1)$. Find $\vec{v} = \overrightarrow{CD}$ and draw it.

Fact: A given vector can be drawn from any initial position. Rephrased: vectors with the same length and the same direction are considered to be the same vector.

Definition: A vector is in **standard position** if it starts at the origin.

Notation: We use square brackets for vectors and round brackets for points.

Example: Let $\vec{u} = [-1, 2]$ and $\vec{v} = [1, 3]$. Find $\vec{u} + \vec{v}$ both algebraically and geometrically.

Example: Graph $\vec{u}, \vec{v}$ and $\vec{u} + \vec{v}$ without a coordinate system.

Example: Let $\vec{v} = [1, 3]$. Graph $2\vec{v}, -\vec{v}$ and $-3\vec{v}$.

Definition: The process of multiplying a vector by a real number is called **scalar multiplication**. It produces a vector that is parallel to the original vector.

Example: Calculate $[2, 6] - [3, 4]$

Example: Place $\vec{u}$ and $\vec{v}$ tail to tail. Find the vector that runs from the head of $\vec{v}$ to the head of $\vec{u}$.

Example: Place $\vec{u}$ and $\vec{v}$ tail to tail. Draw the parallelogram formed by $\vec{u}$ and $\vec{v}$. Label the four diagonals.

Fact: Order doesn't matter when adding vectors. For any vectors $\vec{u}, \vec{v}$ and $\vec{w}$:
$\vec{u} + \vec{v} = \vec{v} + \vec{u}$
$(\vec{u} + \vec{v}) + \vec{w} = (\vec{w} + \vec{u}) + \vec{v}$

Example: Let $\vec{u}, \vec{v}$ and $\vec{w}$ be positioned tail to tail to tail. Show geometrically that
$(\vec{u} + \vec{v}) + \vec{w} = (\vec{w} + \vec{u}) + \vec{v}$

Fact: The above example illustrates that we can write $\vec{u} + \vec{v} + \vec{w}$ without any bracketing.

Definition: Consider the expression: $\vec{v}$ in $\mathbb{R}^n$. This means that $\vec{v}$ has n components, and each component is a real number.

Example: Draw $\vec{v} = [1, 3, 2]$ in $\mathbb{R}^3$.

Definition: The **zero vector** is written $\vec{0}$. Each of its components is zero. The zero vector is useful for algebra.

Example: Write the zero vector in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example: Let $\vec{u}$ be in $\mathbb{R}^2$. Show (prove) that $\vec{u} + (-\vec{u}) = \vec{0}$.

Example: Solve for $\vec{x}$ given that $7\vec{x} - \vec{a} = 3(\vec{a} + 4\vec{x})$.

Definition: Consider the statement:
$\vec{w}$ is a **linear combination** of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ with coefficients -3 and 2.
This means that $\vec{w} = -3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Example: Let $\vec{w} = -3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.
a) Find $\vec{w}$ algebraically.

b) Find $\vec{w}$ geometrically.

Example: Write $\vec{w} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ by graphing.

Example: a) Let $\vec{u}$ be a vector of length 5, in standard position, rotated 30° from the positive x-axis. Find $\vec{u}$ algebraically.

b) Let $\vec{v}$ be a vector of length 7, in standard position, rotated 135° from the positive x-axis. Find $\vec{v}$ algebraically.

Comment: Vectors are often used to represent velocity, acceleration or forces. The vector's direction represents the direction of the velocity/acceleration/force. The vector's length represents the magnitude of the velocity/acceleration/force.

1.2 Length and Angle

Example: Let $\vec{u} = [1, 4, 2, -9]$ and $\vec{v} = [2, 3, -2, -1]$. Calculate the dot product $\vec{u} \cdot \vec{v}$

Example: Calculate:
a) $[1, 5] \cdot [2, -3]$

b) $[1, 5] \cdot [2, -3, 0]$

c) $[u_1, u_2] \cdot [u_1, u_2]$

Fact: Three Properties of the Dot Product
Let $\vec{u}, \vec{v}$ be in $\mathbb{R}^n$. Then:
1) $\vec{u} \cdot \vec{u} \geq 0$
2) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
3) $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$

Example: Break Property 3 into two statements, and decide which is more obvious.

Example: Simplify:
a) $(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$

b) $3\vec{u}\cdot(-2\vec{v}+5\vec{w})$

Definition: The **length** of $\vec{v}$ is written $||\vec{v}||$. If $\vec{v}=[v_1,v_2,\ldots,v_n]$ then $||\vec{v}||=\sqrt{v_1^2+v_2^2+\ldots+v_n^2}$.

Example: Draw a picture to show that in 2D this is the Pythagorean Theorem.

Example: Calculate:
a) $||[1,1,1,-2]||$

b) $||[3,-1]||$

c) $[3,-1]\cdot[3,-1]$

Fact: $\vec{v}\cdot\vec{v}=||\vec{v}||^2$ for all $\vec{v}$

Example: Let $\vec{v} = [v_1, v_2, v_3]$. Simplify $||-3\vec{v}||$.

Fact: $||c\vec{v}|| = |c| \ \ ||\vec{v}||$ for all vectors $\vec{v}$ and real numbers c.

Definition: A **unit vector** is a vector that has length one. **Normalizing** a vector $\vec{v}$ means finding a unit vector in the same direction as $\vec{v}$.

Fact: The following vector has length one and the same direction as $\vec{v}$
(provided that $\vec{v} \neq \vec{0}$):
$\vec{u} = \frac{1}{||\vec{v}||}\vec{v}$

Example: Normalize $\vec{v} = [4, -2, 1]$

Definition: The **distance** between $\vec{a}$ and $\vec{b}$ is written $d(\vec{a}, \vec{b})$. It is calculated by $d(\vec{a}, \vec{b}) = ||\vec{a} - \vec{b}||$

Example: Draw a picture to illustrate the above formula.

Example: Find the distance between $\vec{a} = [2, -1]$ and $\vec{b} = [3, -6]$

Fact: The Triangle Inequality
For all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$: $||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$

Fact: Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^n$. The angle θ between $\vec{u}$ and $\vec{v}$ is defined to be $0^\circ \leq \theta \leq 180^\circ$

Fact: For all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$: $\quad \vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta$

Comment: In $\mathbb{R}^4$ and higher dimensions, this is a definition of θ.

Comment: In the special case where $\vec{u}$ and $\vec{v}$ are unit vectors, $\vec{u} \cdot \vec{v}$ gives the value of $\cos\theta$.

Example: Find the angle between $\vec{u} = [1, -4]$ and $\vec{v} = [2, 3]$

Example: If $0° \leq \theta < 90°$, what is the sign of $\vec{u} \cdot \vec{v}$?
What if $\theta = 90°$?
What if $90° < \theta \leq 180°$?

Definition: Vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if $\vec{u} \cdot \vec{v} = 0$.

Comment: The following statements are equivalent in 2D and 3D:
Vectors $\vec{u}$ and $\vec{v}$ are perpendicular (geometry language)
Vectors $\vec{u}$ and $\vec{v}$ are orthogonal (algebra language)

Comment: In higher dimensions it's more appropriate to use the word **orthogonal** rather than perpendicular.

Definition: The **projection** of $\vec{v}$ onto $\vec{u}$ is written $\text{proj}_{\vec{u}}\vec{v}$. This could be read as the projection onto $\vec{u}$ of $\vec{v}$.

Example: Let's draw a few instances of $\text{proj}_{\vec{u}}\vec{v}$

Fact: $\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2}\vec{u}$

Example: Find $\text{proj}_{\vec{u}}\vec{v}$ for $\vec{u} = [1, 2]$ and $\vec{v} = [1, 3]$

Fact: Given vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$, there is exactly one way to decompose $\vec{v}$ into two vectors that are parallel and perpendicular to $\vec{u}$.

Example: Let $\vec{u} = [1, 1]$ and $\vec{v} = [4, 2]$. Find vectors $\vec{a}$ and $\vec{b}$ so that $\vec{v} = \vec{a} + \vec{b}$, $\vec{a}$ is parallel to $\vec{u}$, and $\vec{b}$ is perpendicular to $\vec{u}$.

1.3 Lines and Planes

Part 1. Lines in $\mathbb{R}^2$

Definition: The **general form** of a line in $\mathbb{R}^2$ is $ax + by = c$

Example: Consider the line $3x + y = 1$. Find two points on the line and graph the line.

Definition: A **normal vector** is orthogonal to a given line. It is written $\vec{n}$. Its components are the coefficients from the general form.

Definition: The **normal form** of a line in $\mathbb{R}^2$ is $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$ where $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{p}$ is the vectorization of any point on the line.

Example: Describe the line $3x + y = 1$ in normal form. Show that expanding normal form gives general form.

Definition: A **direction vector** for a line is $\vec{d} = \overrightarrow{PQ}$, where P and Q are any two points on the line.

Definition: The **vector form** for a line in $\mathbb{R}^2$ is $\vec{x} = \vec{p} + t\vec{d}$, where t represents any real number.

Example: Describe the line $3x + y = 1$ in vector form. Show that as t varies, the line is traced out.

Definition: The **parametric form** for a line in $\mathbb{R}^2$ is:

$$\begin{cases} x = a + bt \\ y = c + dt \end{cases}$$

Example: Describe the line $3x + y = 1$ in parametric form.

Comment: A given line can be described in a specific form in multiple ways, for example $3x + y = 1$ and $6x + 2y = 2$ are general forms for the same line.

Example: Summarize the four forms of a line in $\mathbb{R}^2$

Part 2. Lines in $\mathbb{R}^3$

Example: Consider the line through $P = (2, 1, 12)$ and $Q = (0, -3, 6)$. Describe the line in both vector and parametric form.

Definition: A **plane** is an infinite flat surface.

Fact: $ax + by + cz = d$ is the general form for a plane in $\mathbb{R}^3$.

Comment: General form for a line in $\mathbb{R}^3$ is inconvenient so we will omit it. It would consist of two equations, describing the intersection of two planes.

Comment: Similarly we omit normal form for a line in $\mathbb{R}^3$.

Part 3. Planes in $\mathbb{R}^3$

Example: Consider the plane through $P = (1, -1, 3)$ with normal $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Describe the plane in both normal and general form.

Definition: The **vector form** for a plane in $\mathbb{R}^3$ is $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$ where:
$\vec{u}$ and $\vec{v}$ are nonparallel direction vectors
s and t represent any real numbers

Example: Consider the plane through $P = (6, 0, 0)$, $Q = (0, 6, 0)$ and $R = (0, 0, 3)$. Describe the plane in vector and parametric form.

Example: Summarize the twelve descriptions

Part 4. Geometry Problems

Example: Find the distance between $B = (1, 3, 3)$ and the plane $\mathcal{P} : x + y + 2z = 7$

Example: Find the distance between $B = (1, 1, 0)$ and the line through $A = (0, 1, 2)$ with $\vec{d} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Comment: To find the distance between parallel planes, pick a point on one of the planes. Find the distance between that point and the other plane.

Comment: To find the distance between parallel lines, pick a point on one of the lines. Find the distance between that point and the other line.

Definition: The **angle between planes** is defined as the angle between their normals.

Definition: **Parallel planes** have parallel normals.
Perpendicular planes have perpendicular normals.

1.4 The Cross Product

The cross product $\vec{u} \times \vec{v}$ is defined for $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$.

Example: Let $\vec{u} = [1, 2, 1]$ and $\vec{v} = [3, -1, 4]$. Calculate $\vec{u} \times \vec{v}$.

Example: Let $\vec{u} = [1, 2, 1]$ and $\vec{v} = [3, -1, 4]$. Calculate:
a) $\vec{v} \times \vec{u}$

b) $(\vec{u} \times \vec{v}) \cdot \vec{u}$

Fact: Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^3$. Then:
$\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$ AND
$\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$

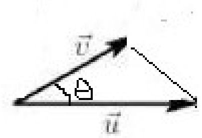
Fact: The vector $\vec{u} \times \vec{v}$ is a normal for the plane containing $\vec{u}$ and $\vec{v}$. The direction of $\vec{u} \times \vec{v}$ is determined by the Right Hand Rule.

Example: Find the general form of the plane through $A = (1, 3, 6)$, $B = (2, 1, 4)$ and $C = (1, -1, 5)$.

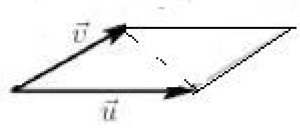
Comment: Recall that $\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta$ for $\vec{u}, \vec{v}$ in $\mathbb{R}^n$.

Fact: If $\vec{u}$ and $\vec{v}$ are in $\mathbb{R}^3$ then $||\vec{u} \times \vec{v}|| = ||\vec{u}||\ ||\vec{v}|| \sin\theta$.

Example: Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^3$. Consider the triangle below.
Show that the area of the triangle is $\frac{1}{2}||\vec{u} \times \vec{v}||$



Fact: Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^3$. Consider the parallelogram below, which can be divided into two triangles with equal area. Then:
Area(triangle)= $\frac{1}{2}||\vec{u} \times \vec{v}||$ AND
Area(parallelogram)= $||\vec{u} \times \vec{v}||$



Example: Find the area of the triangle determined by $\vec{u} = [1, 4, 5]$ and $\vec{v} = [2, 3, 6]$.

Definition: A **matrix** is a rectangular array of numbers. For example, $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

Definition: The **determinant** of a matrix A is written $\det A$ or $|A|$. The determinant is only defined for square matrices.

Fact:
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
AND
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Comment: The second formula is called **cofactor expansion**.

Comment: Notice that the second term in the cofactor expansion has a negative sign.

Example: Compute $\det \begin{bmatrix} 1 & 4 & 6 \\ 2 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix}$

Example: Compute $\begin{vmatrix} -1 & -4 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 8 \end{vmatrix}$

Notation: Let:
$\vec{i} = [1,0,0]$
$\vec{j} = [0,1,0]$
$\vec{k} = [0,0,1]$

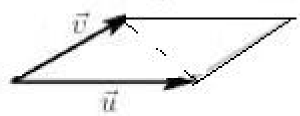
Fact: A second method of calculating the cross product is:
$[u_1, u_2, u_3] \times [v_1, v_2, v_3] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

Example: Calculate $[2,1,3] \times [-6,4,2]$ using the original method.

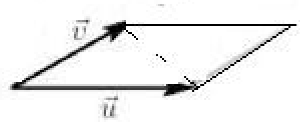
Example: Calculate $[2,1,3] \times [-6,4,2]$ using the second method. Notice why cofactor expansion has a negative sign on the second term.

Fact: Three geometry formulas:

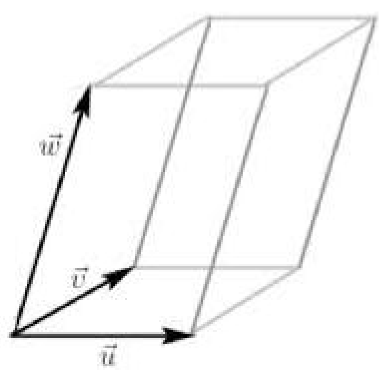
1) Area(parallelogram in $\mathbb{R}^3$)=$||\vec{u} \times \vec{v}||$



2) Area(parallelogram in $\mathbb{R}^2$)= absolute value of det $\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$



3) Volume(parallelepiped in $\mathbb{R}^3$)= absolute value of det $\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$



Example: Find the area of the parallelogram determined by $[1, 6]$ and $[3, 5]$.

Example: Do the vectors $[1, 4, 7], [2, 5, 9]$ and $[1, -2, -3]$ lie in a common plane?

Chapter 2: Systems of Linear Equations

2.1 Linear Systems

Definition: A **linear equation** in $\mathbb{R}^2$ has the form $ax + by = c$, where a, b and c are real numbers.

Definition: A **linear system** in $\mathbb{R}^2$ consists of two or more linear equations. It's often just called a **system**.

Comment: Here's an example of a system:

$$\begin{aligned} 2x + 6y &= -14 \\ -3x + 3y &= -15 \end{aligned}$$

Fact: A system can have: no solution, one unique solution or infinitely-many solutions.

Definition: A system with no solution is called an **inconsistent system**.
A **consistent system** has one solution or infinitely-many solutions. In other words, a consistent system is solvable.

Definition: Consider the system:

$$\begin{aligned} 2x + 6y &= -14 \\ -3x + 3y &= -15 \end{aligned}$$

The matrix $\begin{bmatrix} 2 & 6 \\ -3 & 3 \end{bmatrix}$ is called the **coefficient matrix**.

The matrix $\left[\begin{array}{cc|c} 2 & 6 & -14 \\ -3 & 3 & -15 \end{array}\right]$ is called the **augmented matrix**.

Fact: There are three types of elementary row operations that can be performed on an augmented matrix. These row operations don't change the solution of the system:
1) Swap two rows
2) Multiply or divide a row by a nonzero real number
3) (Current Row) $\pm$ #(Pivot Row)

Example: Solve by elimination:

$$\begin{aligned} 2x + 6y &= -14 \\ -3x + 3y &= -15 \end{aligned}$$

Example: Solve:

$$\begin{aligned} 2x - 3y &= 8 \\ -4x + 6y &= 20 \end{aligned}$$

Fact: A system has no solution if the following type of row appears while performing row operations:
[all zeros | nonzero]

Example: Solve:

$$\begin{aligned} 2x - 3y &= 8 \\ -4x + 6y &= -16 \end{aligned}$$

Example: Solve:

$$\begin{aligned} x &= 5 \\ 2x + 3y &= 4 \\ 3x + 4y &= 7 \end{aligned}$$

Definition: **Back-substitution** is the process of solving a sytem from the bottom equation upwards.

Example: Solve by back-substitution:

$$\begin{aligned} 4x + \ y + \ \ z &= 15 \\ 3y + 5z &= 29 \\ 2z &= 8 \end{aligned}$$

Comment: Most systems can't be solved by back-substitution.

2.2 Solving Systems

Definition: A matrix is in **row-echelon form** (REF) if:
any zero rows are at the bottom AND
the leading nonzero entries of each row move down and right

Comment: The following matrices are in REF:
$\begin{bmatrix} 6 & 0 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 7 \\ 0 & 0 & 0 \end{bmatrix}$

Definition: An augmented matrix is in REF if the coefficient matrix is in REF.

Comment: The following matrices are in REF:
$\left[\begin{array}{ccc|c} 6 & 0 & -1 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right]$ $\quad$ $\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & 4 & 7 & 0 \\ 0 & 0 & 0 & 9 \end{array}\right]$

Definition: One method of solving a system is **Gaussian Elimination**. The augmented matrix is transformed to REF using elementary row operations. The system is then solved by back-substitution.

Example: Solve by Gaussian Elimination:

$$\begin{aligned} x + 2y + z &= 6 \\ 2x + 2y \phantom{{}+ z} &= 8 \\ 3y + z &= 8 \end{aligned}$$

Definition: A matrix is in **reduced row-echelon form** (RREF) if:
the matrix is in REF,
the leading nonzero entry in each row is 1, AND
these leading ones have zeros everywhere else in their columns

Comment: The following matrices are in RREF:
$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comment: The following matrix is in REF but not RREF:
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

Definition: An augmented matrix is in RREF if the coefficient matrix is in RREF.

Comment: The following matrices are in RREF:
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 5 & 0 & 9 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 9 \end{array}\right]$$

Definition: Another method of solving a system is **Gauss-Jordan Elimination**. The augmented matrix is transformed to RREF using elementary row operations. This is typically faster than Gaussian Elimination.

Example: Solve by Gauss-Jordan Elimination:

$$\begin{aligned} x + 2y + 3z &= 7 \\ 3x + 3y + 3z &= 15 \\ 5x + 7y + 9z &= 29 \end{aligned}$$

Example: Solve by Gauss-Jordan Elimination:

$$\begin{aligned} x + \ y - \ \ 6z &= 17 \\ 2x + 2y - \ \ 8z &= 22 \\ 3x + 3y - 14z &= 39 \end{aligned}$$

Example: Solve by Gauss-Jordan Elimination:

$$\begin{aligned} w + \ \ x + 2y + 10z &= 5 \\ x + \ \ y + \ \ \ \ z &= 2 \\ w + 3x + 4y + 12z &= 9 \end{aligned}$$

Example: Find the intersection of the two lines:

$$\vec{x} = \begin{bmatrix} -5 \\ 6 \\ 5 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Example: How many solutions does the following system have?

$$\begin{aligned} x + ky &= 1 \\ kx + \ y &= 1 \end{aligned}$$

Definition: The **rank** of a matrix is the number of nonzero rows in its REF or RREF.

Fact: If a system is consistent then:
rank+(# of parameters in solution)=# of variables

Example: Verify the fact for the following system:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Rephrase the fact in terms of columns of the coefficient matrix.

Comment: Notice that $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a solution to the following system:

$$\begin{aligned} x + 2y &= 0 \\ 3x + 4y &= 0 \end{aligned}$$

Definition: A system whose constants are all zero is called a **homogeneous system**. The solution $\vec{x} = \vec{0}$ is called the **trivial solution**.

Fact: A homogeneous system always has at least one solution: $\vec{x} = \vec{0}$.

Example: Consider a homogeneous system with more variables than equations. How many solutions does the system have?

2.3 Span and Linear Independence

Example: Is $\begin{bmatrix} 8 \\ -10 \end{bmatrix}$ a linear combination of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$?

Example: Is $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ a linear combination of $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$?

Fact: The vector $\vec{b}$ is a linear combination of the columns of matrix A if and only if the system $\left[A|\ \vec{b}\right]$ is consistent.

Definition: The **span** of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ is the set of all linear combinations of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$.

Comment: a) span$(\vec{a}, \vec{b})$= $\{\vec{0},\ 3\vec{a},\ -7\vec{b},\ 2\vec{a}+5\vec{b}, \ldots\}$

b) span$(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$= $\{c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n\}$
where $c_1, c_2, \ldots, c_n$ are any real numbers.

Fact: The zero vector $\vec{0}$ is in span$(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ because $0\vec{u}_1 + 0\vec{u}_2 + \cdots + 0\vec{u}_n = \vec{0}$.

Example: Describe each span geometrically:

a) span($\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \end{bmatrix}$)

b) span($\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$)

c) span($\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}$)

d) span($\begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$)

Example: Find an equation for span($\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$). Give your answer in any form.

Example: a) Show that span($\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$)= $\mathbb{R}^2$.

b) Write $\begin{bmatrix} a \\ b \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Comment: To decide if a system is consistent, reduce it to REF.
To solve a system, reduce it to RREF.

Definition: Given $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, consider solutions to $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$. If the only solution is $c_1 = c_2 = \ldots = c_n = 0$ then the set of vectors is **linearly independent**. If there are solutions other than $c_1 = c_2 = \ldots = c_n = 0$ then the set of vectors is **linearly dependent**.

Comment: The two sentences below mean the same thing:
Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent.
The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent.

Comment: The two sentences below mean the same thing:
Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent.
The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly dependent.

Comment: a) $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \end{bmatrix} \right\}$ is linearly dependent.

b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ are linearly dependent.

c) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are linearly dependent.

Example: Are $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\0\\2\end{bmatrix}$ linearly independent?

Fact: A set of more than n vectors in $\mathbb{R}^n$ is linearly dependent. For example three vectors in $\mathbb{R}^2$ are guaranteed to be linearly dependent.

Example: Let's explore why this fact is true.

Example: Find a linear dependence relationship (linear dependency) involving $\begin{bmatrix} 1 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 30 \end{bmatrix}$. Start by letting $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$.

Example: Find a linear dependence relationship (linear dependency) involving $\begin{bmatrix} 1 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 30 \end{bmatrix}$. Start by putting the vectors into the rows of a matrix.

Comment: Compare the methods used in the last two examples. The first method gives the general solution, while the second method gives one particular solution.

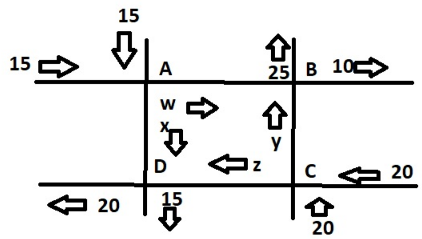
Comment: Preview of Section 3.5:
We'll consider objects like lines or planes through the origin, and find a set of direction vectors containing the minimum number of vectors. This discussion will require knowledge of span and linear independence.

2.4 Applications of Linear Systems

Example: Find the parabola $y = ax^2 + bx + c$ that passes through $(1, 12), (-1, 18)$ and $(2, 30)$.

Example: Balance $NH_3 + O_2 \to N_2 + H_2O$

Example: Consider the following network of one-way streets. The average number of vehicles per hour through intersections A,B,C,D was collected from historical data.
a) Find the flows w, x, y, z.
b) If the solution has a parameter then specify the possible values of the parameter.



Example: Find all possible combinations of 15 coins (nickels, dimes or quarters) that total \$ 2.50.

Chapter 3: Matrices

3.1 Matrix Operations

Definition: The **size** of a matrix is given by (# of rows)×(# of columns).
For example $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is a 2×3 matrix.

Definition: The **entry** of a matrix A is written a_{ij} or $[A]_{ij}$, where i and j are the row index and the column index respectively. For the matrix above $a_{23} = 6$ or $[A]_{23} = 6$.

Definition: A **square** matrix has size $n \times n$.

Definition: An **identity** matrix is square with ones along the main diagonal and zeros elsewhere. It can be written I, or I_n if we want to emphasize its size.

For example $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Definition: A **diagonal** matrix is square and all the entries off the main diagonal are zero.
For example $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ or $D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$.

Example: Let $A = \begin{bmatrix} 1 & 6 & 1 \\ -2 & -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 6 & 9 \end{bmatrix}$. Find:

a) $A + B$

b) $3A$

Comment: $A + B$ is undefined if A and B have different sizes.

Definition: The process of multiplying a matrix by a real number is called **scalar multiplication**.

Example: Let $A = \begin{bmatrix} 1 & 6 & 1 \\ -2 & -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 6 & 9 \end{bmatrix}$. Find $A - 3B$.

Definition: The **transpose** of A, written A^T, interchanges the rows and columns of A. The matrix A is **symmetric** if $A^T = A$.

Example: Calculate the transpose and state if the matrix is symmetric.

a) $A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 6 & 3 \\ 4 & 3 & -1 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 6 & 1 \end{bmatrix}$

Fact: To multiply two matrices:

$$AB = \begin{bmatrix} r_1 \cdot c_1 & r_1 \cdot c_2 & \dots & r_1 \cdot c_n \\ r_2 \cdot c_1 & r_2 \cdot c_2 & \dots & r_2 \cdot c_n \\ \dots & \dots & \dots & \dots \\ r_n \cdot c_1 & r_n \cdot c_2 & \dots & r_n \cdot c_n \end{bmatrix}$$

where r_i is row i of matrix A and c_j is column j of matrix B.

Example: Find AB where $A = \begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix}$.

Example: Let's consider the sizes of A and B in the example above.
Which two numbers must be equal to make AB defined?
Which two numbers predict the size of AB?

Example: Let A be a 2×3 matrix and let B be a 3×1 matrix. Calculate the sizes of AB and BA.

Fact: $AB \neq BA$ in general.

Example: Find BC and CB where $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$.

Example: Expand the following:
$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

Comment: Matrix multiplication was reverse-engineered to solve systems of equations.

Fact: A system of equations can be written as $A\vec{x} = \vec{b}$ where:
A is the coefficient matrix
$\vec{x}$ is the vector of variables, written as a column
$\vec{b}$ is the vector of constants, written as a column

Example: Consider the data below:

	Al	Bob
Test1 Mark	50	60
Test2 Mark	90	80
Exam Mark	75	70

Test1 Weight	Test2 Weight	Exam Weight
0.2	0.2	0.6

Let A be a matrix containing the course marks for the two students. Let B be a matrix containing the weightings of the coursework. Find Al and Bob's final grades using a matrix multiplication.

Definition: The **outer product expansion of** AB is:
$AB = A_1B_1 + A_2B_2 + \ldots + A_nB_n$
where A_i is column i of A and B_j is row j of B.

Comment: Normal matrix multiplication involves **rows** of the first matrix and **columns** of the second matrix.
The outer product expansion involves **columns** of the first matrix and **rows** of the second matrix.

Example: Find the outer product expansion of AB given:
$A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 7 \\ 4 & 2 \end{bmatrix}$.

Example: Confirm the result in the previous example using normal matrix multiplication.

Comment: The outer product expansion will be used further in Section 5.4.

Definition: The expression A^n means multiply A with itself n times. For example:
$A^2 = AA$
$A^3 = A^2A$ or $A^3 = AA^2$ or $A^3 = AAA$

Example: Express A^{12} as the cube of a matrix.

Example: Compute A^2 for $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Fact: Recall that I is the identity matrix. For any matrix A:
$AI = A$ and $IA = A$

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Confirm that $AI = A$ and $IA = A$.

Example: Simplify B^{2018} given that $B^3 = I$.

Definition: Let **O** be the zero matrix. For example **O**$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or **O**$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ etc.

Example: Find a 2×2 matrix A so that $A^2 =$**O** but $A \neq$**O**.

3.2 Matrix Algebra

Example: Is $\begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 0 & 1 \\ 6 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 7 & 2 \end{bmatrix}$?

Example: Find the general form of span$\left(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}\right)$.

Comment: The general form of the span allows us to quickly identify which matrices are in the span. For example, $\begin{bmatrix} 1 & 7 \\ 2 & 30 \end{bmatrix}$ is in the span and $\begin{bmatrix} 1 & 7 \\ 3 & 30 \end{bmatrix}$ is not.

Example: Find the general form of span$\left(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}\right)$.

Example: Are $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$ linearly independent?

Definition: Matrices A and B **commute** if $AB = BA$.

Example: Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 9 \\ 3 & 7 \end{bmatrix}$. Do A and B commute?

We're going to look at six properties of matrices.

Property 1: For any matrices A, B and C with compatible sizes:
$(AB)C = A(BC)$

Example: Verify Property 1 for $A = \begin{bmatrix} 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 6 \end{bmatrix}$.

Property 2: For any matrices A, B and C with compatible sizes:
$A(B+C) = AB + AC$

Example: Verify Property 2 for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Properties 3 and 4: For any matrix A:
$AI = A$ and $IA = A$

Property 5: For any matrices A and B with compatible sizes:
$(A \pm B)^T = A^T \pm B^T$

Example: Break Property 5 into two statements.

Example: Confirm that $(A - B)^T = A^T - B^T$ for $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}$.

Property 6: For any matrices $A_1, A_2, \ldots, A_n$ with compatible sizes:
$(A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T$

Example: Confirm that $(AB)^T = B^T A^T$ for $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}$.

Example: Expand $(A+B)^2$ and simplify.

Example: Show that A^TA is symmetric.

3.3 The Inverse of a Matrix

Definition: An $n \times n$ matrix A is **invertible** if there exists a matrix A^{-1} so that $AA^{-1} = I$ and $A^{-1}A = I$.

Definition: The matrix A^{-1} is called the **inverse of A**.

Example: Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$. Confirm that $A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$.

Comment: 1) Not every square matrix is invertible.
2) If A^{-1} exists then it is unique.
3) $AA^{-1} = I$ if and only if $A^{-1}A = I$, so we only need to check one property.

Definition: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the **determinant of** A is $\det A = ad - bc$.

Fact: If A is a 2×2 matrix then:

$$A^{-1} = \begin{cases} \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, & \text{if } \det A \neq 0 \\ \text{undefined}, & \text{if } \det A = 0 \end{cases}$$

Example: Find A^{-1}:

a) $A = \begin{bmatrix} 1 & -4 \\ 7 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 3 & -2 \\ -9 & 6 \end{bmatrix}$

Fact: If A^{-1} exists then the system of equations $A\vec{x} = \vec{b}$ has a unique solution: $\vec{x} = A^{-1}\vec{b}$.

Example: Let's explore why the above fact is true.

Example: Use A^{-1} to solve:

$$\begin{aligned} 4x - 5y &= -6 \\ -5x + 6y &= 7 \end{aligned}$$

Fact: To find A^{-1} for an $n \times n$ matrix we form the augmented matrix $[A|I]$. We perform row operations to produce I on the left side. The resulting matrix on the right side will be A^{-1}.

Example: Find A^{-1} for $A = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

Comment: By transforming A into I we are "undoing" A. The matrix on the right side will be the matrix that "undoes" A, that is A^{-1}.

Example: Find A^{-1} for $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 6 \\ 2 & 3 & 11 \end{bmatrix}$.

Fact: Suppose a zero row appears on the left side while reducing $[A|I]$. Then A^{-1} does not exist.

We'll look at three properties of A^{-1}.

Property 1: If A^{-1} exists then $(A^{-1})^{-1} = A$.

Property 2: $(A^T)^{-1} = (A^{-1})^T$ for any matrix A.

Example: Verify Property 2 for $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$.

Property 3: For any matrices $A_1, A_2, \ldots, A_n$ with compatible sizes:
$(A_1 A_2 \cdots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1} A_1^{-1}$.

Comment: In particular this means that $(AB)^{-1} = B^{-1}A^{-1}$.

Comment: Let Operation A represent putting on your socks. Let Operation B represent putting on your shoes. To reverse this sequence we have to undo the operations and **reverse the order of operations**. We could express this in matrix terms as $(AB)^{-1} = B^{-1}A^{-1}$.

Comment: Consider Property 3 with all n matrices equal to A. The statement becomes $(A^n)^{-1} = (A^{-1})^n$. This means we can write A^{-n} without confusion.

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find A^{-2}.

Example: Let A, B and X all be invertible $n \times n$ matrices. Solve for X given $(AX)^{-1} = BA$.

Definition: An **elementary matrix** represents a row operation.
To identify which operation, consider how I has been transformed. For example:

$E = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ represents $2R_1$

$E = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$ represents -$4R_2$

$E = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ represents $R_2 + 3R_1$

$E = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$ represents $R_1 - 5R_2$

$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents $R_1 \leftrightarrow R_2$

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ represents $R_2 \leftrightarrow R_3$

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ represents $R_2 + 6R_3$

Example: State the row operation that is represented by the elementary matrix. Then find the inverse matrix.

a) $E_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

b) $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c) $E_3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Fact: An elementary matrix acts on the **left** of a matrix. When an elementary matrix is multiplied on the left of A, it performs the associated row operation on A. For example:
$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ c & d \end{bmatrix}$.

Example: Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Write A and A^{-1} as a product of elementary matrices.

Example: Let $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$. Write A and A^{-1} as a product of elementary matrices.

The Fundamental Theorem of Invertible Matrices
Let A be an $n \times n$ matrix. The following statements are equivalent:
a) A is invertible.
b) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d) The RREF of A is I.
e) A is a product of elementary matrices.

Comment: Consider the Fundamental Theorem of Invertible Matrices. For a given $n \times n$ matrix, the five statements are **all true** or **all false**.

Example: Consider the Fundamental Theorem of Invertible Matrices. Which of the five statements are true for A?

a) $A = \begin{bmatrix} 1 & 4 \\ 6 & 9 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

3.4 LU Factorization

Definition: An **upper triangular matrix** is a square matrix with zeros below the main diagonal. An example is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

Definition: A **lower triangular matrix** is a square matrix with zeros above the main diagonal. An example is $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$.

Definition: A **unit lower triangular matrix** is lower triangular and has ones on the main diagonal. An example is $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}$.

Definition: The LU Factorization of a square matrix A is $A = LU$, where L is a unit lower triangular matrix and U is an upper triangular matrix.

Comment: Here is an LU Factorization:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 4 & 3 \\ 8 & 10 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

Example: Solve the system below using the LU Factorization on the previous page.

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 4 & 3 \\ 8 & 10 & 13 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$$

Fact: To find the LU Factorization of a matrix A:
Transform A to REF using only: (current row)-k(pivot row).
The matrix L has the k-values in the appropriate positions.
The matrix U is the REF.

Fact: The matrix A has an LU Factorization if and only if no row swaps are required to transform A to REF.

Example: Find the LU Factorization of $\begin{bmatrix} 2 & 1 & 1 \\ 4 & 4 & 3 \\ 8 & 10 & 13 \end{bmatrix}$

Example: Let's explore why the method to find the LU Factorization works.

Example: Find the LU Factorization of A and use it to solve:

$$\begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$$

Example Continued...

3.5 Subspaces and Basis

Definition: A **subspace of** $\mathbb{R}^n$ is the span of one or more vectors in $\mathbb{R}^n$.

Comment: a) A line through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

b) A line through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

c) A plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

Example: Is the following set of vectors a subspace of $\mathbb{R}^3$?

$$S = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid \ 3x + 4y + z = 0 \}$$

Example: Is the following set of vectors a subspace of $\mathbb{R}^3$?

$S = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid \; z = x + 1\}$

Example: Is the following set of vectors a subspace of $\mathbb{R}^2$?

$S = \{ \begin{bmatrix} x \\ y \end{bmatrix} \mid \; y = 0\}$

Let's define three subspaces associated with a matrix A.

Definition: The **rowspace of** A is the span of the rows of A, written row(A).
The **columnspace of** A is the span of the columns of A, written col(A).
The **nullspace of** A is $\{\vec{x}| \quad A\vec{x} = \vec{0}\}$, written null$(A)$.

Example: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$.

a) Is $\begin{bmatrix} 6 \\ 10 \end{bmatrix}$ in col(A)?

b) Is $[1, 2, 5]$ in row(A)?

c) Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in null(A)?

Definition: A set of vectors $\mathcal{B}$ is a **basis for a subspace** S if:
span($\mathcal{B}$)=S and $\mathcal{B}$ is linearly independent.

Comment: Let's rephrase this. A set $\mathcal{B}$ is a basis for a subspace S if:
$\mathcal{B}$ is a set of direction vectors for S containing the minimum number of vectors.

Comment: a) $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ is a basis for $\mathbb{R}^2$.

b) $\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}\}$ is a basis for $\mathbb{R}^2$.

c) $\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \end{bmatrix}\}$ is **not** a basis for $\mathbb{R}^2$.

d) $\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}\}$ is **not** a basis for $\mathbb{R}^2$.

e) $\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$ is **not** a basis for $\mathbb{R}^2$.

Example: Let $A = \begin{bmatrix} 2 & 3 & 7 \\ 4 & 7 & 10 \\ 8 & 17 & 8 \end{bmatrix}$. Find a basis for:

a) row(A)

b) col(A)

Comment: In general, performing a row operation changes the columnspace of a matrix. We cannot use the nonzero columns of the REF/RREF to form a basis for col(A).

Example: Let $A = \begin{bmatrix} 2 & 3 & 7 \\ 4 & 7 & 10 \\ 8 & 17 & 8 \end{bmatrix}$. Find a basis for row$(A)$ consisting of rows of A.

Note: This is different from part a) of the previous example, because that answer was not phrased in terms of rows of A.

Example: Let $A = \begin{bmatrix} 1 & 4 & 6 \\ 2 & 8 & 12 \end{bmatrix}$. Find a basis for null$(A)$.

Example: Find a basis for span$\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\6\end{bmatrix}, \begin{bmatrix}1\\5\\24\end{bmatrix}\right)$.

Definition: Given a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$,
the **coordinate vector of $\vec{v}$ with respect to $\mathcal{B}$** is

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix} \text{ where } \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n.$$

Example: Find $[\vec{v}]_{\mathcal{B}}$ for $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \}$ and $\vec{v} = \begin{bmatrix} 5 \\ 15 \\ 28 \end{bmatrix}$.

Definition: The **dimension** of a subspace S is the number of vectors in a basis for S. It's written $\dim(S)$.

Comment: a) The standard basis for $\mathbb{R}^3 = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$. Therefore $\dim \mathbb{R}^3 = 3$.

b) $\dim \mathbb{R}^n = n$
c) dim(plane through the origin in $\mathbb{R}^n$)$= 2$
d) dim(line through the origin in $\mathbb{R}^n$)=1

Definition: The **rank** of a matrix is the number of nonzero rows in its REF or RREF.

Comment: For any matrix A: rank(A)=dim(row(A))=dim(col(A)).

Definition: The **nullity** of a matrix A is the number of parameters in the solution to $A\vec{x} = \vec{0}$. In other words, nullity(A)=dim(null(A)).

Example: Let $A = \begin{bmatrix} 1 & 5 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. Find rank($A$) and nullity($A$).

Fact: For any matrix A: rank(A)+nullity(A)= number of columns in A.

Example: Let's phrase this fact in terms of the columns of A.

The Fundamental Theorem of Invertible Matrices
Let A be an $n \times n$ matrix. The following statements are equivalent:
a) A is invertible.
b) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d) The RREF of A is I.
e) A is a product of elementary matrices.
f) $\text{rank}(A) = n$.
g) $\text{nullity}(A) = 0$.
h) The columns of A are linearly independent.
i) The span of the columns of A is $\mathbb{R}^n$.
j) The columns of A form a basis for $\mathbb{R}^n$.
k) The rows of A are linearly independent.
l) The span of the rows of A is $\mathbb{R}^n$.
m) The rows of A form a basis for $\mathbb{R}^n$.
n) $\det A \neq 0$.
o) 0 is not an eigenvalue of A.

Comment: Consider the Fundamental Theorem of Invertible Matrices. For a given $n \times n$ matrix, the fifteen statements are **all true** or **all false**.

Example: Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

3.6 Linear Transformations

Definition: A **transformation** is an operation that turns a vector into another vector.

Example: The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates a vector by 90° counterclockwise. Graph the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ before and after the transformation.

Definition: The vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is called the **image of** $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **under** T .

We can write $T(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ or $T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Definition: The **matrix transformation** T_A multiplies a vector on the left by matrix A. In other words, $T_A(\vec{x}) = A\vec{x}$.

Example: a) Let $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & -3 \end{bmatrix}$. Find $T_A(\begin{bmatrix} x \\ y \\ z \end{bmatrix})$.

b) Find A given $T_A(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x + y \\ x - y \\ 3x + 3y \end{bmatrix}$.

Definition: A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if:
$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ and
$T(c\vec{u}) = cT(\vec{u})$ for all real numbers c and all vectors $\vec{u}$ in $\mathbb{R}^n$.

Fact: The transformation T is linear if and only if T is a matrix transformation.

Example: Show that T is linear given $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} y \\ x \end{bmatrix}$.

Example: Show that T is not linear given $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} y \\ 1 + x \end{bmatrix}$.

Definition: The **standard matrix for** T is the matrix that performs T. It's written $[T]$.

Fact: Let $T : \mathbb{R}^2 \to \mathbb{R}^m$ be a linear transformation. To calculate $[T]$:

Place $T(\begin{bmatrix}1\\0\end{bmatrix})$ in the first column and place $T(\begin{bmatrix}0\\1\end{bmatrix})$ in the second column.

In other words, $[T] = \left[T(\begin{bmatrix}1\\0\end{bmatrix}) \;\middle|\; T(\begin{bmatrix}0\\1\end{bmatrix}) \right]$.

Fact: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then:

$$[T] = \left[T(\begin{bmatrix}1\\0\\0\\\dots\\0\end{bmatrix}) \;\middle|\; T(\begin{bmatrix}0\\1\\0\\\dots\\0\end{bmatrix}) \;\middle|\; \dots \;\middle|\; T(\begin{bmatrix}0\\0\\0\\\dots\\1\end{bmatrix}) \right].$$

Comment: The formula for $[T]$ works because T is linear.

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that reflects a vector in the y-axis. Find:

a) $[T]$

b) $T(\begin{bmatrix} x \\ y \end{bmatrix})$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that reflects a vector in the line $y = x$. Find $[T]$.

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates a vector by angle θ (counterclockwise). Find $[T]$.

Example: Rotate $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ by $30°$ clockwise.

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that projects a vector on the line l through the origin with direction vector $\vec{d} = \begin{bmatrix} a \\ b \end{bmatrix}$. Find $[T]$.

Comment: It's recommended that you know the following two standard matrices:

Rotation by angle θ (counterclockwise): $\quad [T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Projection onto the line $\vec{x} = t \begin{bmatrix} a \\ b \end{bmatrix}$: $\quad [T] = \frac{1}{a^2+b^2} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$

Definition: Suppose we apply T_1 then T_2 to $\vec{x}$. This is a **composition** of transformations. It can be written $T_2(T_1(\vec{x}))$ or $(T_2 \circ T_1)(\vec{x})$.
We calculate it as $[T_2]\,[T_1]\,\vec{x}$.

Example: Let $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} 2x \\ -y \end{bmatrix}$. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation by $45°$. Find $[S \circ T]$.

Definition: Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$.
The **inverse of** T is a transformation $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ such that:
$T^{-1}(T(\vec{x})) = \vec{x}$ and $T(T^{-1}(\vec{x})) = \vec{x}$ for all vectors $\vec{x}$ in $\mathbb{R}^n$.

Comment: Note that T^{-1} is only defined when $[T]$ is invertible.

Fact: The standard matrix for T^{-1} is the inverse of the standard matrix for T.

Example: Rewrite this fact using appropriate notation.

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation by $-30°$. Find $[T^{-1}]$.

Example: Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$. Suppose:

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $T(\vec{v}_1) = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$,

$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $T(\vec{v}_2) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, and

$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $T(\vec{v}_3) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Find $T(\begin{bmatrix} 7 \\ 3 \\ 6 \end{bmatrix})$.

Chapter 4: Eigenvalues and Eigenvectors

4.1 Eigenvalues and Eigenvectors, 2×2 Matrices

Definition: Let A be an $n \times n$ matrix. Suppose $A\vec{x} = \lambda\vec{x}$ for some vector $\vec{x} \neq \vec{0}$ and some real number λ. Then λ is an **eigenvalue of** A and $\vec{x}$ is an **eigenvector of** A.

Example: Show that $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}$.

Comment: We say that $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of A corresponding to eigenvalue $\lambda = 4$.

Comment: Note that $A\vec{0} = \lambda\vec{0}$ is trivial. Therefore the zero vector is never considered to be an eigenvector.

Example: Find all eigenvectors of $A = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix}$ corresponding to $\lambda = 6$.

Fact: To find all the eigenvectors corresponding to eigenvalue λ:
Solve the system $[A - \lambda I \mid \vec{0}]$. Remember to exclude $\vec{x} = \vec{0}$.

Definition: The **eigenspace** E_λ is the set of all eigenvectors of A corresponding to eigenvalue λ, plus the zero vector. It's a subspace of $\mathbb{R}^n$.

Example: Find a basis for E_3 given $A = \begin{bmatrix} 4 & 1 & -2 \\ -3 & 0 & 6 \\ 2 & 2 & -1 \end{bmatrix}$.

Example: Find a basis for E_0 given $A = \begin{bmatrix} 4 & 1 & -3 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

Fact: Let B be an $n \times n$ matrix. The system $B\vec{x} = \vec{0}$ has nontrivial solutions exactly when $\det B = 0$. (This follows from the Fundamental Theorem of Invertible Matrices).

Fact: To find all the eigenvalues of A: Solve the equation $\det(A - \lambda I) = 0$.

Example: Let's understand why solving $\det(A - \lambda I) = 0$ gives the eigenvalues.

Example: Find all the eigenvalues of $A = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}$.

Example: Find a basis for E_{-6} given $A = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}$.

Comment:

To find eigenvalues: Solve the equation $\det(A - \lambda I) = 0$.

To find eigenvectors: Solve the system $[A - \lambda I \mid \vec{0}]$. Remember to exclude $\vec{x} = \vec{0}$.

Example: Let $A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$. Find the eigenvectors and eigenvalues geometrically.

4.2 Determinants

Comment: Recall that the determinant of A is written $\det A$ or $|A|$. It's only defined for square matrices.

Fact: The cofactor expansion from Section 1.4 generalizes according to the following rules:

We can expand along any row or column.

The sign associated with each term follows the checkerboard pattern: $\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$.

Example: Find $\det A$ by cofactor expansion along the second column. Calculate it again by cofactor expansion along the third row. Let $A = \begin{bmatrix} 4 & 1 & 6 \\ 1 & 2 & 3 \\ 6 & 0 & 7 \end{bmatrix}$.

Example: Calculate $|A|$ for $A = \begin{bmatrix} 1 & 6 & 2 & 3 \\ 0 & 0 & 0 & 4 \\ 2 & 1 & 1 & 6 \\ 2 & 0 & 5 & 7 \end{bmatrix}$.

Example: In this example we'll illustrate the **Quick Method** for 3×3 Determinants. Calculate $\det A$ using the Quick Method. Let $A = \begin{bmatrix} 1 & 4 & 9 \\ 2 & -2 & 6 \\ 1 & 0 & 4 \end{bmatrix}$.

Comment: The Quick Method only applies for 3×3 matrices.

Fact: If A is upper triangular, lower triangular or diagonal then $\det A$ is the product of the diagonal entries. For example $\det \begin{bmatrix} 2 & 9 & 13 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = -8$.

Example: Let's understand why by calculating $\det \begin{bmatrix} 2 & 9 & 13 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$.

Fact: How Row Operations Change the Determinant:

$R_i \pm kR_j$ does not change the determinant.

$R_i \leftrightarrow R_j$ changes the sign of the determinant.

We can factor any row, for example $\det \begin{bmatrix} 3 & 6 \\ 1 & 5 \end{bmatrix} = 3 \det \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}$.

Example: Calculate the determinant by reducing the matrix to REF.

Let $A = \begin{bmatrix} 1 & -2 & 1 & 9 \\ 2 & 1 & 3 & 3 \\ 3 & 1 & 4 & 5 \\ 0 & 1 & 1 & 6 \end{bmatrix}$.

Comment: In general $\det A \neq \det(\text{REF of } A)$.

Fact: An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

Fact: Properties of $\det A$:

1) $\det A^{-1} = \frac{1}{\det A}$ (if $\det A \neq 0$)

2) $\det AB = \det A \cdot \det B$

3) $\det kA = k^n \det A$ (where A is $n \times n$)

4) $\det A^T = \det A$

Comment: To illustrate Property 3:
$\det \begin{bmatrix} 7a & 7b \\ 7c & 7d \end{bmatrix} = 7^2 \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\det \begin{bmatrix} 5a & 5b & 5c \\ 5d & 5e & 5f \\ 5g & 5h & 5i \end{bmatrix} = 5^3 \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Comment: Note that $\det(A+B) \neq \det A + \det B$ in general.

Example: Let $\det A \neq 0$. Prove Property 1.

Fact: Cramer's Rule
Let A be an $n \times n$ matrix. When $\det A \neq 0$, the system $A\vec{x} = \vec{b}$ has a unique solution:
i-th variable= $\frac{|A_i|}{|A|}$
where $A_i = A$ with the i-th column replaced by $\vec{b}$.

Example: Solve using Cramer's Rule:

$$\begin{aligned} 2x + 3y + 2z &= -11 \\ 3x + 5z &= 23 \\ 4x + y + z &= 1 \end{aligned}$$

Definition: A **cofactor** is the signed determinant in the cofactor expansion that's associated with a matrix entry. It's written C_{ij}.

The sign is given by the checkerboard pattern: $\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$.

Example: Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 5 \end{bmatrix}$.
Calculate the cofactors C_{11}, C_{12} and C_{32}.

Definition: The **cofactor matrix** is the matrix whose entries are the cofactors of A.

Example: Find the cofactor matrix for $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 5 \end{bmatrix}$.

Definition: The **adjoint of** A is the transpose of the cofactor matrix. It's written $\text{adj}(A)$.

Fact: For an $n \times n$ matrix with $|A| \neq 0$:
$A^{-1} = \frac{1}{|A|}\,\text{adj}(A)$.

Example: Let $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 2 \\ 3 & -3 & 4 \end{bmatrix}$. Find A^{-1} using the adjoint formula.

Example: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find A^{-1} using the adjoint formula.

4.3 Eigenvalues and Eigenvectors, $n \times n$ Matrices

Example: Find all the eigenvalues of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & 3 \\ 0 & 0 & 7 \end{bmatrix}$.

Fact: The eigenvalues of an upper triangular, lower triangular or diagonal matrix are the diagonal entries.

Integer Roots Theorem
If a polynomial has integer coefficients and the leading coefficient is 1 then any integer roots divide the constant.

Example: Find all the eigenvalues of $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$.

Example Continued...

Definition: The **characteristic equation** of A is $|A - \lambda I| = 0$.
The **algebraic multiplicity** of an eigenvalue λ_i is the exponent on $(\lambda_i - \lambda)$ in the characteristic equation.
The **geometric multiplicity** of an eigenvalue is the number of basis vectors in the corresponding eigenspace.

Example: Let A have characteristic equation $(7-\lambda)^3(9-\lambda)^2 = 0$. A basis for E_7 consists of one vector. A basis for E_9 consists of two vectors. Find the eigenvalues of A and state their algebraic multiplicities and their geometric multiplicities.

Fact: For each eigenvalue:
$1 \leq$ geometric multiplicity $\leq$ algebraic multiplicity

Comment: If a matrix has
(geometric multiplicity) = (algebraic multiplicity) for all its eigenvalues
then the matrix has a nice property.
We'll see the details in Section 4.4.

We'll look at five properties of eigenvalues.

Property 1: A is invertible if and only if 0 is not an eigenvalue of A.

Example: Prove Property 1.

Property 2: If A is invertible and $A\vec{x} = \lambda\vec{x}$ then $\vec{x}$ is an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.

Example: Prove Property 2.

Property 3: Let n be a non-negative integer. If $A\vec{x} = \lambda\vec{x}$ then $A^n\vec{x} = \lambda^n\vec{x}$.

Example: Prove Property 3.

Property 4: If $A\vec{x} = \lambda\vec{x}$ then $\vec{x}$ is an eigenvector of $A + kI$ with eigenvalue $\lambda + k$.

Example: Prove Property 4.

Property 5: Let n be a non-negative integer.
Suppose A has eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m$ corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. Then:
$A^n(c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_m\vec{x}_m) = c_1\lambda_1^n\vec{x}_1 + c_2\lambda_2^n\vec{x}_2 + \ldots + c_m\lambda_m^n\vec{x}_m$.

Comment: This is a generalization of Property 3.
Note that the coefficients are preserved.

Example: Suppose A has:
eigenvalue $\lambda_1 = -2$ corresponding to eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and

eigenvalue $\lambda_2 = 3$ corresponding to eigenvector $\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Calculate $A^3 \begin{bmatrix} 11 \\ 2 \end{bmatrix}$.

Example: Suppose A has the eigenvalue 3 corresponding to the eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

List one eigenvector and one eigenvalue for each of the following matrices: $A^{-1}, A^4, A + 2I$.

4.4 Diagonalization

Definition: An $n \times n$ matrix A is **diagonalizable** if there exist an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

Fact: To find P we find a basis for each eigenspace of A. The basis vectors go into the columns of P. The matrix D has the eigenvalues on the diagonal, in the same order as P.

Example: Let $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find P and D that diagonalize A.

Example Continued...

Fact: A is diagonalizable if and only if:
geometric multiplicity=algebraic multiplicity for all eigenvalues of A.

Example: Diagonalize $A = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}$ (if possible).

Example: Let $A = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}$. Find the characteristic equation, the algebraic multiplicity of $\lambda = 4$ and the geometric multiplicity of $\lambda = 4$. Explain, in terms of algebraic and geometric multiplicity, why A can't be diagonalized.

Fact: Let n be a positive integer. If D is diagonal then D^n is diagonal, with n-th powers on the diagonal.

Example: Calculate $\begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}^2$.

Fact: Let n be a positive integer. If $P^{-1}AP = D$ then $A^n = PD^nP^{-1}$.

Example: Prove the fact above.

Example: $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizes A to produce $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Find A^k, where k is a positive integer.

Example: Application of A^n and eigenvectors. This example **will not be tested**.
Consider a company with 1000 machines.
a) Suppose a working machine has a 99% probability of working tomorrow. Suppose a broken machine has a 50% probability of being broken tomorrow. Write down the probability matrix, A.

b) Suppose all machines are working today. Write down the initial state vector, $\vec{v}$.

c) How many machines will be working or broken tomorrow?

d) How many machines will be working or broken two days from now?

e) How many machines will be working or broken three days from now?

f) How many machines will be working or broken n days from now, where n is a non-negative integer?

g) What initial state vector $\vec{v}$ would have $A\vec{v} = \vec{v}$? This is called the **steady-state vector** because the state after one day is the same as the initial state.

Chapter 5: Orthogonality

5.1 Orthogonality

Definition: An **orthogonal set** is a set of two or more vectors such that any two of the vectors are orthogonal.

Example: Verify that $\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \}$ is an orthogonal set.

Definition: To **normalize** a vector means to find a unit vector in the same direction.

Example: Normalize $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Definition: An **orthonormal set** is an orthogonal set in which all vectors have length 1. For example, the following is an orthonormal set:

$\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \}$.

Fact: A set of n nonzero orthogonal vectors in $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$.

Comment: This implies that a set of n nonzero orthonormal vectors in $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$.

Example: Find an orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ for $\mathbb{R}^3$ such that:
$\vec{u}_1$ is parallel to $[2, 0, 1]$ and $\vec{u}_2$ is parallel to $[1, 3, -2]$.

Fact: Suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthogonal basis for $\mathbb{R}^n$.
For any vector $\vec{w}$ in $\mathbb{R}^n$:

$\vec{w} = \text{proj}_{\vec{v}_1}\vec{w} + \text{proj}_{\vec{v}_2}\vec{w} + \ldots + \text{proj}_{\vec{v}_n}\vec{w}$

Example: Draw a sketch to show that $\vec{w} \neq \text{proj}_{\vec{v}_1}\vec{w} + \text{proj}_{\vec{v}_2}\vec{w} + \ldots + \text{proj}_{\vec{v}_n}\vec{w}$ if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is not orthogonal.

Example: $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \}$ is an orthogonal basis for $\mathbb{R}^3$. Write $\vec{w} = \begin{bmatrix} 5 \\ 0 \\ 9 \end{bmatrix}$ as a linear combination of the basis vectors.

Definition: An **orthogonal matrix** Q is an $n \times n$ matrix whose columns form an orthonormal set. For example, the following matrix is orthogonal:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Fact: A square matrix Q is orthogonal if and only if $Q^T Q = I$.

Example: Verify that $Q^T Q = I$ for $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Fact: If Q is orthogonal then $Q^{-1} = Q^T$.

Example: Prove the fact above.

Example: Let Q be an orthogonal matrix. Show that Q^{-1} is orthogonal.

Example: Determine all values of x, y and z so that $\begin{bmatrix} \frac{1}{2} & y \\ x & z \end{bmatrix}$ is an orthogonal matrix.

5.2 Orthogonal Complements and Projections

Throughout Chapter 5, W will represent a subspace of $\mathbb{R}^n$. Rephrased: W is the span of one or more vectors in $\mathbb{R}^n$.

Definition: The **orthogonal complement of** W is:
$W^\perp = \{\vec{v}$ in $\mathbb{R}^n$ such that $\vec{v} \cdot \vec{w} = 0$ for all $\vec{w}$ in $W\}$.
$W^\perp$ is pronounced "W perp".

Example: Let $W = \text{span}(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix})$. Find $W^\perp$.

Recall that the **dimension** of a subspace W is the number of vectors in a basis for W.

Three Facts about $W^\perp$

For any subspace W of $\mathbb{R}^n$:

1) $\dim W + \dim W^\perp = n$

2) $W \cap W^\perp = \{\vec{0}\}$

3) $(W^\perp)^\perp = W$

Example: Let $W = \text{span}(\begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix})$. Find the dimension of W and $W^\perp$.

Example: Let $W = \{ \begin{bmatrix} x \\ y \end{bmatrix}$ such that $3x + y = 0\}$. Find a basis for W and for $W^{\perp}$.

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 6 \end{bmatrix}$ and let $W = \text{row}(A)$. Find a basis for $W^\perp$.

Fact: For any matrix A, $[\text{row}(A)]^\perp = \text{null}(A)$.

Example: Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & 0 \end{bmatrix}$ and let $W = \text{col}(A)$. Find a basis for $W^{\perp}$.

Fact: For any matrix A, $[\text{col}(A)]^{\perp} = \text{null}(A^T)$.

Definition: Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$.
The **orthogonal projection of** $\vec{v}$ onto W is:

$\text{proj}_W\vec{v} = \text{proj}_{\vec{w}_1}\vec{v} + \text{proj}_{\vec{w}_2}\vec{v} + \ldots + \text{proj}_{\vec{w}_k}\vec{v}$.

Comment: This formula only applies when the basis for W is **orthogonal**.

Example: Let W be a plane through the origin in $\mathbb{R}^3$. Let $\vec{v}$ be a vector in $\mathbb{R}^3$ that does not lie in W. Sketch W, $\vec{v}$ and $\text{proj}_W\vec{v}$.

Example: W has orthogonal basis $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \}$.

Find the orthogonal projection of $\vec{v} = \begin{bmatrix} 1 \\ 5 \\ -3 \\ 7 \end{bmatrix}$ onto W.

Definition: The **orthogonal decomposition** of $\vec{v}$ with respect to W is:

$\vec{v} = \text{proj}_W\vec{v} + \text{perp}_W\vec{v}$

where $\text{proj}_W\vec{v}$ is in W and $\text{perp}_W\vec{v}$ is in $W^\perp$.

Example: Let W be a plane through the origin in $\mathbb{R}^3$. Let $\vec{v}$ be a vector in $\mathbb{R}^3$ that does not lie in W. Sketch W, $\vec{v}$, $\text{proj}_W\vec{v}$ and $\text{perp}_W\vec{v}$.

Example: W has orthogonal basis $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \}$.

Find the orthogonal decomposition of $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ with respect to W.

5.3 The Gram-Schmidt Procedure

Example: Let $W = \text{span}(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -4 \\ -2 \end{bmatrix})$. Find an orthogonal basis for W.

Example Continued...

Comment: This procedure is called **Gram-Schmidt Orthogonalization**.

Example: Modify the basis above to create an orthonormal basis for W.

Example: Find an orthogonal basis for $\mathbb{R}^3$ containing $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$.

Example Continued...

Definition: Let A be a matrix with linearly independent columns.
The **QR Factorization of** A is:
$A = QR$ where Q is an orthogonal matrix and R is upper triangular.

Example: Let $A = QR$ for an orthogonal matrix Q. Show that $R = Q^T A$.

Fact: Let $A = QR$ for an orthogonal matrix Q.
To find Q: Apply Gram-Schmidt to the columns of A, and normalize.
Then $R = Q^T A$.

Example: Find Q and R for $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

Example Continued...

Example: Approximating the eigenvalues of A. **This example will not be tested**.

Consider the following procedure:
Find $A = Q_0R_0$.
Let $A_1 = R_0Q_0$ then find $A_1 = Q_1R_1$.
Let $A_2 = R_1Q_1$ then find $A_2 = Q_2R_2$ etc.
Each matrix A_k has the same eigenvalues as A.
As $k \to \infty$, A_k becomes upper triangular.

Suppose we start with matrix A and produce $A_4 = \begin{bmatrix} 1.98 & 2.52 \\ 0.03 & 7.01 \end{bmatrix}$.

a) Does A_4 have the same eigenvalues as A?

b) Is A_4 approximately upper triangular?

c) Estimate the eigenvalues of A.

5.4 Orthogonal Diagonalization

Recall that if Q is orthogonal then $Q^{-1} = Q^T$.

Definition: An $n \times n$ matrix A is **orthogonally diagonalizable** if there exist an orthogonal matrix Q and a diagonal matrix D so that $Q^T AQ = D$.

Fact: Let A be an $n \times n$ matrix. The matrix A is orthogonally diagonalizable if and only if A is symmetric.

Example: Is A orthogonally diagonalizable?

a) $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

b) $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

Example: The matrix $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ has eigenvalues $\lambda = 4$ and $\lambda = 7$.
Find Q that orthogonally diagonalizes A.

Example Continued...

We're going to recap the outer product expansion of AB from Section 3.1.

Example: Find $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ using the outer product expansion.

Example: Find $\begin{bmatrix} -1 & 9 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ using the outer product expansion.

Definition: Let A be a symmetric $n \times n$ matrix.
Let $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n$ be <u>orthonormal</u> eigenvectors written as <u>columns</u>.
Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues.
The **spectral decomposition of** A is:
$A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \ldots + \lambda_n \vec{q}_n \vec{q}_n^{\,T}$

Example: Find a 3×3 matrix A with eigenvalues $\lambda = 2$ and $\lambda = 3$ so that:
$E_2 = \text{span}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix})$ and $E_3 = \text{span}(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix})$.

Example Continued...

Example: Suppose $Q^T AQ = D$. Solve for A then use the outer product expansion to derive the spectral decomposition.

Appendix: Other Topics

7.3 Least Squares Approximation

Recall that a system $A\vec{x} = \vec{b}$ may be inconsistent.

Definition: Given an approximate solution $\vec{s}$, the **error vector** is $\vec{b} - A\vec{s}$ and the **error** is $||\vec{b} - A\vec{s}||$.

Definition: The **least squares solution** $\vec{x}^*$ is the approximate solution with the minimum error.

Comment: Recall that $||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$. The terminology **least squares solution** emphasizes that we're making the length of the error vector as small as possible.

Fact: The **least squares solution** to a system $A\vec{x} = \vec{b}$ is $\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$.

Comment: We'll assume that the columns of A are linearly independent so that $(A^TA)^{-1}$ exists.

Example: The system $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ is inconsistent.

Find the least squares solution $\vec{x}^*$.

Example: Calculate the error for $\vec{x}^*$ above. What can you say about the error for any other vector $\vec{x}$?

Example: Find the best-fit line $y = a_0 + a_1 x$.
The best-fit line is also called the **least squares regression line**.

x	y
0	4
1	1
2	0

Example: Find the best-fit parabola through:

x	y
1	1
2	-2
3	3
4	4

Example: Find the best-fit curve $P = Ce^{kt}$ through:

t	P
0	5
1	8
3	12

Example Continued...

Recall that $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is in col(A).
This follows from Sections 2.3 and 3.5.

Recall that null(A^T)=[col$(A)]^\perp$. We saw this in Section 5.2.

Example: Derive the formula for $\vec{x}^*$ by considering an inconsistent system $A\vec{x} = \vec{b}$.

Example Continued...

Complex Numbers

Definition: Let i be the **imaginary number** such that $i^2 = -1$.
If a and b are real numbers then $z = a + bi$ is a **complex number**.

Comment: The symbol i is sometimes written j. You may feel free to use either notation.

Example: Let $z_1 = -2 + 6i$ and $z_2 = 4 + 5i$. Calculate:

a) $-7z_1$

b) $z_1 + z_2$

c) $z_1 - z_2$

d) $z_1 z_2$

Definition: The **complex conjugate** of $z = a + bi$ is $\overline{z} = a - bi$.

Example: Let $z = a + bi$. Show that $z\overline{z} = a^2 + b^2$.

Example: Let $z_1 = 4 + 9i$ and $z_2 = -3 + 5i$. Calculate:

a) $\frac{1}{z_1}$

b) $\frac{z_1}{z_2}$

Definition: The **length** of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.
The **principal argument** of $z = a + bi$ is the angle $\theta = \tan^{-1}(\frac{b}{a})$ $(+\pi?)$
We decide whether to add π or not based on the graph of z.

Example: Let $z = -1 + 2i$. Graph z then calculate $|z|$ and θ.

Example: Show that $z = |z|[\cos\theta + i\sin\theta]$.

Definition: The **rectangular form** of a complex number is $z = a + bi$.
The **polar form** of a complex number is $z = |z|[\cos\theta + i\sin\theta]$.

Example: Express $z = -1 + 8i$ in polar form.

Fact: Multiplication and Division in Polar Form

Let $z_1 = |z_1|[\cos\theta_1 + i\sin\theta_1]$ and $z_2 = |z_2|[\cos\theta_2 + i\sin\theta_2]$.

Then $z_1 z_2 = |z_1|\;|z_2|[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$ and

$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$.

Comment: When multiplying in polar form: multiply the lengths and add the angles. When dividing in polar form: divide the lengths and subtract the angles.

Example: Let $z_1 = 9 + 3\sqrt{3}i$ and $z_2 = 4\sqrt{3} - 12i$.
Find $\frac{z_1}{z_2}$ by converting to polar form.

Example Continued...

Fact: De Moivre's Formula
Let n be a positive integer.
If $z = |z|[\cos\theta + i\sin\theta]$ then $z^n = |z|^n[\cos(n\theta) + i\sin(n\theta)]$.

Example: Find $(1-i)^{21}$.

Example: Calculate i^0, i^1, i^2, i^3, i^4 and i^5.

Fact: Let n be a non-negative integer. Then:
$i^{4n} = 1, \quad i^{4n+1} = i, \quad i^{4n+2} = -1$ and $i^{4n+3} = -i$.

Example: Simplify i^{271}.

Example: Recall that:
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
Show that $e^{i\theta} = \cos\theta + i\sin\theta$.

Example: Derive the **most beautiful equation in mathematics** by subbing $\theta = \pi$ into the equation $e^{i\theta} = \cos\theta + i\sin\theta$.

Definition: The **rectangular form** of a complex number is $z = a + bi$.
The **polar form** of a complex number is $z = |z|[\cos\theta + i\sin\theta]$.
The **exponential form** of a complex number is $z = |z|e^{i\theta}$.

Now we'll look at complex eigenvalues and eigenvectors.

Example: Let $A = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix}$.

a) Find the eigenvalues.

b) Find a basis for one of the eigenspaces.

c) Find a basis for the other eigenspace.

Fact: A complex number has 2 square roots, 3 cube roots, and n different n-th roots for integers $n \geq 2$.

Fact: $z = |z|[\cos\theta + i\sin\theta]$ has n different n-th roots for integers $n \geq 2$:
$z^{\frac{1}{n}} = |z|^{\frac{1}{n}}[\cos\frac{\theta+2\pi\alpha}{n} + i\sin\frac{\theta+2\pi\alpha}{n}]$ for $\alpha = 0, 1, \ldots, n-1$.

Example: Find all the cube roots of -1.

Example: Solve $x^2 - 2i = 0$.